RASHTRIYA SANSKRIT SANSTHAN, NEW DELHI

(Kendriya Sanskrit Vidyapeetha, Tirupati Series No. 29)

SIDDHĀNTA ŚIROMAṆI

OF

BHĀSKARĀCĀRYA

English Exposition and Annotation in the
light and language of modern Astronomy

By

Dr. D. ARKASOMAYAJI

Ex-Principal, D. N. R. College, Bhimavaram, A.P.,
Ex-Reader in Astronomy, Kendriya Sanskrit Vidyapeetha, Tirupati,
Recipient of President's Award.

KENDRIYA SANSKRIT VIDYAPEETHA

TIRUPATI

1980

First Edition 1980

© All Rights reserved

Price Rs.

Published for the
RASHTRIYA SAṄSKRIT SANSTHAN
Ministry of Education, S. W. & Culture.,
Government of India,
New Delhi

By

Dr. M. D. BALASUBRAHMANYAM,
Principal, Kendriya Sanskrit Vidyapeetha,
TIRUPATI (A.P.)

Printed by the Rathnam Press, Madras-600 001.

राष्ट्रियसंस्कृतसंस्थानम्, नवदेहली

[केन्द्रीयसंस्कृतविद्यापीठग्रन्थमाला, संख्या-29]

भास्कराचार्यविरचितः

सिद्धान्तशिरोमणिः

डा० धूलिपाल० अर्कसोमयाजिकृतया आधुनिकखगोलशास्त्रानुसारिण्या
आङ्गलभाषाव्याख्यया अलंकृतः

संपादक :
डा० धूलिपाल० अर्कसोमयाजी

केन्द्रीयसंस्कृतविद्यापीठम्, तिरुपतिः

1980

प्रथमप्रकाशनम् 1980

मूल्यम्—रू.

सर्वेऽधिकाराः राष्ट्रियसंस्कृतसंस्थानेन स्वायत्तीकृताः

शिक्षासमाजकल्याणमन्त्रालयान्तर्गतदेहलीस्थ-राष्ट्रियसंस्कृतसंस्थानस्य कृते
तिरुपतिस्थकेन्द्रीयसंस्कृतविद्यापीठस्य प्राचार्येण
डा. मल्लादि द. बालसुब्रह्मण्यशर्मणा प्रकाशितः
रत्न प्रेस्, मद्रास् इत्येतैः मुद्रितश्च ।

DEDICATION AT THE FEET OF LORD VENKATEŚVARA

समर्पणम्

श्लो. 1. त्वत्पादाब्जदिदृक्षया हिमगिरिप्रस्यन्दिगङ्गाझरी

सङ्काशप्रवहज्जनावळिरहोरात्रं च जोघुष्यते ।

गोविन्देति विमुक्तकण्ठमसकृत् श्री वेङ्कटेशप्रभो !

त्वत्पादाब्जसमर्पिता कृतिरियं भक्त्या प्रसूनायताम् ॥

श्लो. 2. यः सिद्धान्तशिरोमणि र्विजयते श्रीभास्करार्यस्य सः

व्याख्यातो बहुभिस्तथाऽपि मयका व्याख्यायतेऽसौ पुनः ।

व्याख्येयं मणिदीधिति र्विरचिता यादृक्श्रमेण प्रभो !

जानीषे भगवन् ! त्वमेव तमिमां त्वत्पादयो रर्पये ॥

श्लो. 3. पुराचार्यैः प्रोक्तं यदपि च नवीनै स्तदखिलम्

विचार्यैव व्याख्या व्यरचि भगवन् ! भास्करकृतेः ।

यदीयं व्याख्या मे सहृदयबुधाह्लादनकरी

तदाऽहं धन्यः स्यामियमपि कृषिर्मे सफलिति ॥

DEDICATION

English Translation of the Sanskrit verses

1. Oh ! Lord Venkateśvara ! I offer thee this work of mine as a flower at Thy feet, who art being approached day and night by throngs of people in a flow like that of the Ganga, jumping down the heights of the Himalayas, crying out full-throated one of Thy names ' Govinda ' in shrill imploring voices, surging out from the depths of agonizing hearts !

2. The great work named Siddhānta Śiromaṇi of Bhāskarācārya, was indeed commented upon by a host of scholars both ancient and modern. Yet, I not a great scholar make bold to comment once again. Oh ! Lord Venkateśvara ! *Thou alone Knowest* and no mortal does, what stress and strain I underwent while working on this commentary. Hence I dedicate this at Thy feet alone not to any mortal, however great he may be !

3. This commentary has been written by me carefully understanding the depths of both the ancient and modern systems of astronomy. If this could invoke the pleasure of well –meaning scholars, then I deem myself fortunate, and that my toil will have been well rewarded.

श्लो. 4 यो ग्रन्थो महता श्रमेण लिखितः प्रत्नैर्विशेषैर्युतः

यत्सारोऽपि विपश्चितां सुमनसां सूक्ष्मेक्षिकागोचरः ।

यः कश्चिद्यदि पिष्टपेषणमिदं स्यादित्यबुध्द्वा वदेत्

यद्वा किञ्चिदवेक्ष्य सोऽत्र निहितान् भावान् कथं ज्ञास्यति ॥

श्लो. 5. यः सिद्धान्तशिरोमणिं न पठितः श्रीभास्करार्यस्य य

श्चाऽऽत्मानं लघुपुस्तकेक्षणपरः चेत्पण्डितंमन्यते ।

रङ्गत्तुङ्गतरङ्गितं हिमगिरिप्रस्यन्दि गङ्गाजलम्

हित्वा तेन सुपङ्किलं किल जलं स्नातुं समाश्रीयते ॥

व्याख्याता
डा. धूळिपाळोद्वा अर्कसोमयाजी

4. This work indeed sucked the very blood out of me. It contains many a new detail unnoticed hitherto. Only unbiased scholars could understand the essence of this work. If somebody pronounces unknowingly that this is all a repetition of what has been already written, simply having turned out a few pages, how could he know where I showed new aspects of the genius of Bhāskara ?

5. He, who does not study the Siddhānta Śiromaṇi of Bhāskarācārya, and feels that he is a scholar reading a few other sub-standard books on Hindu astronomy, does verily go to bathe in a dirty pond, ignoring the holy Ganga, jumping down the heights of the Himalayas in surging and dancing billows !

D. ARKASOMAYAJI

व्याख्यात्रभिजनः

श्लो. 1. आन्ध्रेषु प्रथितं विराजति पुरं गोदावरीतीरगम्

श्रीमद्राजमहेन्द्रमित्यभिहितं यस्यान्तिके भ्राजते ।

ग्रामः श्रीबलिचेरु नाम विबुधैर्देदीप्यमानस्सदा

देवब्राह्मणयायजूकविलसद्वंशे जनिस्तत्र मे ॥

श्लो. 2. माता मे मङ्गमाम्बा सततमपि पतिं सेवमाना च दुर्गाम्

देवीं नित्यं भजन्ती निजतनुमनयत्सेव्यसेवां चरन्ती ।

विद्वान् बापय्यनामा श्रुतिविहितपथे सञ्चरन् मे पिताऽऽसीत्

मातापित्रो स्तयोर्मे पदभजनमहो बाल्य एव व्यरंसीत् ॥

3. श्रीमद्वेङ्कट रामाख्यमग्रजं कर्मठं स्तुवे ।

प्राविक्षं वेदवेदाङ्गवाङ्मयं यत्पदान्तिके ॥

4. द्वितीय मग्रजं वन्दे सुब्रह्मण्याह्वयं ततः ।

देवब्राह्मणपूजायां जीवितं यस्य यापितम् ॥

5. वेदवेदाङ्गविद्यायां बाल्ये हन्त ! प्रवेशितः ।

जीविकायै ततो हूणभाषामध्यापितोऽभवम् ॥

BIOGRAPHICAL

1. In the state known as the Andhra Pradesh, there is a famous city called Rajahmundry on the banks of the holy Godāvarī. Not far away from this City, there is a village known Velicheru, teeming with Vedic and Sanskrit scholars. There I had my birth in a family reputed for a religious conduct.

2. My mother, by name, Mangamāmbā led her life both in the worship of my father, and her favourite Deity Kanaka Durgā. My father, a Vedic scholar conducted his life on a rigorous Vedic path. Alas! It was not given to me to serve them for long!

3. I bow to my eldest brother Śri Venkata Rāma who leads a rigorously religious life, and who it was, that initiated me into a study of the Vedic and Sanskrit lore.

4. I bow to my next elder brother by name Subrahmaṇya Somayāji who has passed away having spent his life in worshipping God and godly Brahmins.

5. Alas! Though I was initiated into the Vedic and Sanskrit lore in my boyhood, I was later distracted into the secular English education only to eke out my livelihood, as if this is the Summum Bonum of Life.

6. मादृशाः कति वा विप्रकुलोद्भूताः श्रुतिस्मृतीः ।
हित्वा हन्ताङ्ग्लविद्यायां प्रविष्टाः कलिकालतः ॥

7. श्रीनिवासमहं सेवे योऽत्र पर्वतमूर्धनि ।
येन स्वपादसेवाया आनीतः स्वपदान्तिकम् ॥

8. यत्सेवा च दिवारात्रं धर्मपत्न्या कृता मम ।
आवामत्र निवासार्थमानयत् स्वगृहादिव ॥

9. स्वस्ति तस्मै भवेद्राज्ञे रक्षितुं यो व्यवस्यति ।
पारावारगभीरं तद्वेदवेदाङ्गवाङ्मयम् ॥

व्याख्याता
धूळिपाळोह्वा अर्कसोमयाजी

6. How many like me, born in Brahmin families, are being proselytized into the English system of education burying in the Bay of Bengal all that is grand in the Vedic and Sanskrit lore.

7. I raise my hands in supplication to Lord Venkateśvara, who took His abode on the Seven Hills as though to look down upon this mundane and irreligious world. It was He that has brought me here, to worship Him residing at His very feet.

8. I feel it was the deep devotion of my wife to Lord Venkateśvara that has brought us here, dislodging us from our home and hearth.

9. May those administrators flourish, who exert and strive to save the ocean—like Vedic lore from the yawning mouth of Oblivion !

D. ARKASOMAYAJI

FOREWORD

Bhāskarācārya, the Second (c. 1100 A.D.) was one of the greatest mathematicians and astronomers of the World. He anticipated modern theory on the convention of signs (minus × minus = plus) and KEPLER'S method of determining the surface and the volume of a sphere. Six hundred years before the calculus of LEIBNITZ and NEWTON, Bhāskara worked at the differential coefficient. He raised and solved the problem —

$$67\,x^2 + 1 = y^2$$

—a problem which FERMAT resolved after 500 years. The credit goes to Bhāskarācārya for having delineated the image of *Jyotiṣa* in its proper contours and graces.

The *Siddhāntaśiromaṇi* is often said to be the *magnum opus* of Bhāskarācārya. This monumental treatise consists of four parts : (1) *Līlāvatī* (arithmetic), (2) *bījagaṇita-* (algebra), (3) *Golādhyāya-* (Trigonometry including spherical trigonometry) and (4) *Grahagaṇita-* (Planetary motion).

This work was edited with the *Vāsanābhāṣya-* of the author by Bapu Deva SASTRI. Muralidhar JHA brought out two commentaries—the *Vāsanāvārttika-* of Nṛsiṁha (1621 A.D.) and the *Marīci* of Munīśvara (1635 A.D.) on the first Chapter of the *Gaṇitādhyāya* (1917). Girija Prasad DVIVEDI's commentaries in Sanskrit and Hindi (volumes I & II) appeared in 1911 and 1926.

Bapu Deva SASTRI and WILKINSON published an English translation of the text in 1861. Yet the mathematical aspect of the *Siddhāntaśiromaṇi* has remained a *terra incognita* to students of Hindu Astronomy. While *Phalita-Jyotiṣa*-continues to be studied in traditional Sanskrit institutions, the *Gaṇita*-side of Astronomy is reduced to a secondary position.

Realising that specialists in the twin fields of Mathematics and Astronomy have been diminishing day by day, Kendriya Sanskrit Vidyapeetha at Tirupati has started a project entitled, "Coordination of Sanskrit and Ancient Indian Sciences". Under this scheme, Dr. Arka Somayaji has now come forward to give an exposition and annotation of the *Siddhāntaśiromaṇi* in simple English based on the language of modern Astronomy for the benefit of the students and scholars interested in the *Khagola-śāstram*.

Dr. Arka Somayaji is an eminent scholar in Mathematics and Dynamic or Spherical Astronomy. He hails from a family of "Siddhāntin-s". To whet his appetite in learning the mnemonical methodology for compiling an almanac, he studied Mathematics and Jyotiṣa (including Spherical Astronomy). He learnt the *Taittirīya-saṁhitā* under his brother, Dhulipala Venkatarama Avadhani of Rajahmundry. He wrote a thesis on "*A Critical Study of the Ancient Hindu Astronomy*", which was published in 1972. He has to his credit eight works including the *Jyotirvijñānam* (1964) and two Sanskrit poetical compositions (*Brahmāñjali* and the *Hanumat-vijaya-*). He won the President's award in 1974.

In order that critical editions of rare and valuable texts on *Jyotiṣa* be brought out, the Tirupati Vidyapeetha appointed Dr. Arka Somayaji as Reader in Hindu Astronomy. Accordingly the Vidyapeetha has undertaken the publication of his English annotation of the *Siddhāntaśiromaṇi.* I trust this treatise will go a long way not only to project India's image in the World of Mathematics and Astronomy, but also inspire scholars from the transoceanic distance to listen to the jungle roar of the ancient Indian Wisdom.

KENDRIYA SANSKRIT VIDYAPEETHA
TIRUPATI
October 21, 1980

M. D. BALASUBRAHMANYAM
Principal

PREFACE

From the Mathematician's point of view, Bhāskarācārya's Siddhānta Śiromaṇi contains all that was beautiful in the Ancient Hindu Astronomy. I am aware that a translation of this work into English was done long ago by M. M. Bapudeva Śastry and Wilkinson; but I did not have the good fortune of having a copy in my hands all these years. I did go through the translation once long ago, but it gave me the impression that all the beauties there that appeal to a Mathematician were not brought out fully. There are, however, a good number of Sanskrit commentaries both ancient and modern but even they, in my humble opinion, have not done full justice to the elucidation of Bhāskara's mathematical genius. Further misinterpretations are not infrequent in some of those books, as will be pointed out in the course of this book.

What has sponsored me to undertake to write an English commentary, (I may add that a Sanskrit commentary also has been written by me on this work and has been awaiting printing) is essentially that innumerable modern professors of Mathematics complain many a time that there is no such a presentation of Siddhānta Śiromaṇi in English as will enable them to assess the Mathematical content of it in the light and language of modern Astronomy. Hence, I have sought to produce a fresh commentary, which I hope will meet the desire of such professors and students of Mathematics. I may add here that this work of mine seeks to present only the *mathematical side* of Siddhānta Śiromaṇi. It is no history of Hindu Astronomy, where the originality of the Ancient Hindu Astronomers is sought to be evaluated. I may confess that I am no historian.

In the course of this book, I shall have occasion to quote from most of the Astronomers of ancient India like Aryabhaṭa I, Varāhamihira, Lalla, Aryabhaṭa II, Brahmagupta, Bhāskara I, Munjāla, Vateśvara and Śrīpati not to speak of some others, whose books are available in print or manuscript.

One thought that lurks in my mind, I make bold to present in this preface. I say 'I make bold' because, there are some historians of Hindu Astronomy, who are too ready to attack me if I claim that there must have been not a primitive astronomical activity in ancient India prior to Aryabhaṭa I. The reason for their attack is that prior to Aryabhaṭa's work, only one crude Vedānga Jyotiṣa has come to light. Also some of these historians have a strong impression that the galaxy of Hindu Astronomers ranging from Aryabhaṭa derived an incentive from a foreign source especially the Greek. If such historians agree to keep an open mind, I make bold to present my thought as follows. In my humble opinion there must have been considerable astronomical activity in ancient India even prior to Aryabhaṭa. This is borne out by the following expressions of Aryabhaṭa and others. Aryabhaṭa says[1] that he dived into the then extant astronomical lore, which got mixed up with mathematical and non-mathematical (mythological or otherwise) knowledge, and by his intellect and the grace of his Goddess, brought-out the truly mathematical. Varāhamihira[2] says that he was codifying the then extant five Siddhāntas, out of

---

1. सदसत्ज्ञानसंमुद्रात् समुद्धृतं देवताप्रसादेन
सद्ज्ञानोत्तमरत्नं मया निमग्नं स्वमतिनावा — Verse 49—Goḷapāda

2. पूर्वाचार्यमतेभ्यो यद्यत् श्रेष्ठं लघुस्फुटं बीजम्
तत्तदिहाऽविकलमहं रहस्यमभ्युद्यतो वक्तुम् — Verse 2—Ch. I.

पौलिशकृतः स्फुटोऽसौ तस्यासन्नस्तु रोमकप्रोक्तः
स्पष्टतरः सावित्रः परिशेषौ दूरविभ्रष्टौ — Verse 4—Ch. I.

which the 'Sāvitra' was more accurate. Brahmagupta[3] says, again, that he was giving a clear presentation of the ancient Brahma Siddhānta which got obsolete by a long lapse of time. In the wake of these statements and a number of others, is it not right to construe that there did exist some ancient astronomical texts which are lost to us. If, as asserted by a host of modern interpreters, we think that there was only the crude Vedāngajyotiṣa, before Aryabhaṭa, does it not tantamount to saying that these three great astronomers (leave others) were impostors, who, having derived their knowledge from a foreign source, simply claimed that there were existent before them, Saura, Brāhma and some other Siddhāntas which dealt with Graha-gaṇita. It is uncharitable to say that three rational astronomers were sueh impostors. So, we must conclude that there did exist some astronomical activity, which we have no right to call primitive like the Vedāngajyotiṣa. Some might think that there might be existing some other texts in between the times of Vedāngajyotiṣa and Aryabha but that at the time of the Vedāngajyotiṣa[4] (roughly 1180 B.C.) astronomy in India was that crude. Even this conclusion need not be

---

"I am going to tell clearly what has been, the best, what has been kept secret and briefly-worded in a nut shell from out of the works of ancient Acāryas".

"The Siddhānta of pauliśa is allright, the Romaka is very nearly the same, but the Sūryasiddhānta is more accurate, whereas the two others Vāsiṣtha and paitāmaha are crude".

3. ब्रह्मोक्तं ग्रहगणितं महता कालेन यत् खिलीभूतम्
अभिधीयते स्फुटं तत् जिष्णुसुतब्रह्मगुप्तेन ॥ Verse 2—Ch. I.

4. As per statement आश्लेषार्धात् दक्षिणमुत्तरमयनं रवेर्धनिष्ठाद्यम् of Varāhamihira quoting the meaning of the verse.
प्रपद्येते श्रविष्ठादौ सूर्याचन्द्रमसावुदक्

सार्पार्धे दक्षिणा......Verse 7—Vedāngajyotiṣa, the vernal equinoctial point was at the beginning of Dhaniṣtha, which means that the time was 1180 B.C. approximately.

correct, for the simple reason that, even today, crude works exist side by side with advanced works. The simple fact that a Vedāngajyantiṣa belonging to 1180 B.C. has been unearthed and nothing else, is not a complete proof that there did not exist more advanced texts some where else in such a big country like India especially when locomotion or transport was difficult.

Just one more thought, I place before the learned historians, which is pertinent to this context. According to geology, biology and some other similar modern sciences, the first man came into the cosmic picture some millions of years ago. If that be so, how is it that we say that man remained stupid all these years and happened suddenly to blossom into a genius one fine morning round about Eighteen fifties (1850 A.D.) whereafter came all Scientific discoveries in a series, as if by the waving of a magic wand? It could not be so, as if, we are the chosen few of God. Civilizations must have been there, which got buried in the bosom of the earth, as has been revealed by the Mohenzadaro excavations. Hence we are not right in saying that the ancient Hindu civilization was so primitive at 1150 B.C. as is reflected in the Vedānga Jyotiṣa. 'This is another and more important reason that this author makes bold to place before the historians as to why he (the author) does have faith in the statements of Aryabhaṭa, Varāhamihira and Brahmagupta who said that they were peacing out the knowledge contained in the then extant works signifying at the same time that many more books were lost even to them.

In this commentary of mine, I have chosen to keep the commentary away from the Sanskrit text and Bhāskara's own Vāsanā Bhāṣya, for, otherwise the book grows bulkily unwieldy. I have chosen to keep silent over passages which do not call for a mathematical elucidation. Here and there I have chosen to present what modern astronomy presents in some particular contexts, so that,

mathematics students who do not happen to study modern astronomy may have a better perspective of the ancient Hindu astronomy, presented in Juxtaposition to the corresponding modern treatment. However, I do not propose to enter into the intricacies of modern astronomy which are not called for to elucidate the text.

Before concluding, it is my sacred duty to thank the Raṣṭrīya Sanskrit Samsthan, under the Ministry of Education, Social Welfare and Culture, Government of India, for having accepted my work under the publication series of K. S. Vidyapeetha, Tirupati and the late-lamented Dr. M. Ananthasayanam Ayyangar, former Chairman of KSV and Ex-Governor of Bihar, who encouraged me in my studies on Hindu Astronomy. Dr. M. D. Balasubrahmanyam deserves my thanks for the encouraging interest he has shown in the publication of this annotation,

D. ARKASOMAYAJI

CONTENTS

		Page
	Foreword ...	xv
	Preface ...	xix
1.	सिद्धान्तशिरोमणौ गणिताध्याये मध्याधिकारे कालमानाध्यायः ...	1
2.	Madhyādhikāra – Section II Bhagaṇādhyāya ...	18
3.	Grahānayanādhyāya ...	29
4.	Madhyādhikāra - Kakṣādhyāya ...	45
5.	The Adhyāya known as Pratyabda Śuddhi in Madhyādhikāra ...	53
6.	The Section known as Adhimāsādi-nirṇaya in Ch. I ...	70
7.	The Section Bhū-Paridhi-Mānādikara —Circumference of the Earth ...	83
8.	Spaṣṭādhikāra - Rectification of Planets ...	102
9.	The Tripraśnādhikāra ...	223
10.	Parvasambhavādhikāra ...	331
11.	Lunar Eclipses ...	346
12.	Grahacchāyādhikāra ...	449
13.	Grahodayāstādhikāra ...	471
14.	Sṛṅgonnatyadhikāra ...	482
15.	Grahayutyadhikāra ...	501
16.	Pātādhyāya ...	521
	Appendix ...	545

ओं श्रीमहागणपतये नमः, श्रीगुरुभ्यो नमः

सिद्धान्तशिरोमणौ गणिताध्याये मध्याधिकारे कालमानाध्यायः

Verse 1. May the Sun at once give expression to our tongue in meaning ful words-that Sun, who rises to protect this world, whose duty it is to expel darkness, who is reported to be the Spouse of the lotus-creeper, who purges out the sins of all those that supplicate him, and on whose rising take place Vedic sacrifices and thereby the gods in heaven headed by Indra get feasted.

Commentary (*Comm.*). Needs no Commentary.

Verse 2. Excels that blessed Brahmagupta, the son of Jishnu, who is hailed as the crest-jewel of Mathematicians; excel also those like Varāhamihira who were the authors of well known works, who were adepts in reasoning, and u age of beautiful expression and on a study of whose works, one like me even of lesser intellect, will be able to produce monumental works.

Comm. Needless to comment. However, we may state one thing. Bhaskara's quoting the name of Brahmagupta at the very outset, and that too with reverence, informs us that he is going to accept the Āgama of Brahmagupta in preference to that of others like Aryabhata. This means that he is going to adopt the number of revolutions of planets in a Yuga and such other astronomical constants as were adopted by Brahmagupta. This point will be clarified later.

Verse 3 This blessed author Bhaskara is now writing the work named Siddhānta-Siromani, the crest-jewel of all astronomical works, for the pleasure of good minded astronomers, after having bowed to the lotus-feet of his

father, from whom he has derived his knowledge; this work, containing good metres, will be easy to understand, besides being flawless and clear, and will enable intelligent readers to develop their ability to understand things.

Comm. Not neeessary.

Verse 4. Ancient astronomers did write, of course, works abounding in intelligent expression; nonetheless, this work is started to give expression to some lacunae in their works. I am going to make amends for the deficiencies of the older works and these improvements will be found here and there in their respective places; So, I beseech the good-minded mathematicians to go through this entire work of mine also (for, otherwise, they may not locate my own contribution).

Comm. Not necessary.

Verse 5. May the good people be pleased with the particulars of my contribution! May the ill-minded people also derive pleasure out of ridiculing me with ignorance unable to understand my contribution!

Comm. Not necessary.

Verse 6. A Siddhānta work ie an astronomical treatise is such a one which deals with the various measures of time ranging from a Truti (to be explained shortly) upto the duration of a Kalpa which culminates in a deluge; planetary theory, arithmatical computations as well as algebraical processes, Questions with respect to intricate ideas and their answers, location of the earth, the stars and the planets, and description and usage of instruments.

Comm. Not necessary.

Verse 7. Though one knows astrology and that part of the science of Jyotisha which is known as Samhita (and which deals with various subjects like *Muhurtas* i.e. auspi-

cious moments to be prescribed for various functions, Desarishtas ie Calamitous occurences to the countries etc.) which form a part of the Science, he cannot answer so many intricate problems pertaining to Artronomy. Such a person, who does not know the astronomical part of the science, which abounds in innunerable reasonings, is one like a king depicted in a drawing, or a lion fast tied to a pole.

Comm. Not necessary.

Verse 8. The Science of Jyautisha without Astronomy, is like a king's army without roaring elephants though excelling in horses etc; is like a garden without mango trees, or like a lake without water, or again like a lady parted with her newly married lover.

Comm. Not necessary.

Verse 9. The Vedic lore prescribes Sacrifices to be performed; these sacrifices are based upon a knowledge of appropriate time to perform them. This science of astronomy gives a knowledge of time; hence it has been reckoned as one of the six Vedāngas or limbs of the Veda.

Comm. Not necessary.

Verse 10. (Out of the six Vedangas) The science of grammar is like the face of the person of the Veda, the science of Jyautisha takes the place of the eyes, the Nirukta that of the ears; the Kalpa that of the hands; the Siksha that of the nose and the Chandas the place of the feet.

Comm. Not necessary.

Verse 11. This science of Jyautisha being depicted as the very eyes of the Person of the Veda, so it has been acclaimed as the most important of the six Angas or limbs of the Veda, in as much as, even if a person be endowed

with limbs like the ears, nose etc, if he be devoid of vision, he could not do anything.

Comm. Not necessary.

Verse 12. (Since this Science of Astronomy has been declared as the most important of the Vedangas) hence this has to be studied by the Dwijas (ie Brahmins Kshatriyas and the Vaishyas who form the three higher castes), also because it is sacred, secret and the best discipline. By so doing they would acquire Dharma, Artha, Kāma as well as fame (Life is depicted as having a four-fold purpose out of which, Dharma, Artha and Kāma form the first trio, Moksha being the ultimate goal of life).

Comm. Not necessary.

Verses 13 and 14. The creator having created the stellar circle along with the planets, placed the latter at the beginning of the circle, put them in constant revolution, at the same time putting the extreme two stars (on either side) in a fixed position.

Comm. Bhaskara's own commentary Vāsanā Bhasha under this verse mentions the following points, which are to be noted. The twenty seven stars known as Aswini, Bharani etc. occupy positions roughly at equal distances along the Zodiac, arranged from west to east. The planets were all placed in the beginning of the stellar circle in such a way that they were in a straight line the moon, Mercury, Venus, Sun, Mars, Jupiter and Saturn occupying consecutive positions from the earth in increasing distances, not of uniform measure. The circle of stars known as the Zodiac lies far behind all the planets. There is a wind known as pravaha which keeps the entire Zodiac, as well as the planets below going round and round in the Westerly direction. At the same time, the planets while participating in this westerly motion, have themselves individual motion towards the east. Two stars, are placed

one at the north-pole and the other at the south, and they are fixed[1]. The diurnal motion due to pravaha is far greater than the individual motion of the planets among the stars in a direction from west to east ie the direction opposite to that of diurnal motion.

In this context, it is worth-noting that Aryabhata I mentioned the diurnal rotation of the earth in the verse नुलोमगतिः...'' Bhaskara must have been aware of this; yet, in spite of his rational outlook in all matters, misguided himself in this respect (vide verse 3 Madhyagativāsanā-Goladhyaya), apparently for fear of tradition.

Verse 15. The first Mahayuga, the first year, the first day of the bright half of the first month named Madhu,[2] all of them began simultaneously at the Sun-rise[3] at Lankā on Sun-day, at the beginning of the first Kalpa which marked the beginning of creation.

Comm. Though, in the Commentary before, Bhaskara gave expression to the fact that Time is eternal with no beginning and no end, herein he mentions the point of Time when creation commenced. So, at the back of his mind, the concept of Time arose only at the beginning of creation, whereas before creation, as well as after deluge the आत्यन्तिकप्रलय, there was and would be neither the concept of Time nor space. In other words, both space and Time are manifest only after creation, and get extinguished after deluge.

Verses 16, 17 and 18. The unit of Time named Tatpara is 1/30th of what is known as Nimeṣa or the time taken during the fall of an eyelid; One-hundredth of a Tatpara is known as a Tṛti ie the time taken to pierce a lotus-leaf with the finest needle. Eighteen Nimeṣas are equal to a Kāṣthā. Thirty Kāṣthas are equal to a Kalā, thirty Kalās are equal to one sidereal ghatī. Two

ghatīs make a Kshana and thirty Kshanas make a sidereal day. Thirty sidereal days are equal to a sidereal month and twelve sidereal months make a sidereal year (not the sidereal solar year). The Zodiac, divided into twelve rāsīs, and 360 degrees, a degree divided into 60 minutes of arc and a minute divided into 60 seconds of arc all correspond to the year and its successive divisions.

Comm. In the Commentary under these verses, Bhaskara gives further details of division of time as follows. The time taken to pronounce a guru, i.e. double the time of pronouncing a short vowel, is one-tenth of a Prāna, which is the time required by a healthy person to inhale and exhale once. Six Prānas make one Vighatī and sixty ghatis make one sidereal day. It may be clearly noted here that this sidereal month consisting of thirty sidereal days is not the sidereal month that is the time taken by the Moon to go round the Zodiac, nor the sidereal year defined above is the time that the Sun takes to go round the Zodiac once in his apparent annual motion. To distinguish these latter divisions of time, we shall use the nomenclature sidereal lunar month and sidereal solar year. There are further other divisions of time which will be later elucidated.

Verses 19, 20. The time taken by the Sun to complete one revolution with respect to the stars goes by the name 'The sidereal solar year'. This will be a day for the gods and demons. The time that elapses between two consecutive new moons or conjunctions of the Moon with the Sun is called a Chāndra-māsa or a lunar month or simply a lunation. This again is the day of the Pitṛs or the Manes.

The time that elapses between two consecutive Sun rises at a place is termed the Sāvana day or civil day. This is called Saura-Sāvana day and it is also the day of the earth.

The sidereal day is the time taken by the stars to go round the earth once. It is called Nākshatra-dina.

Comm. In Hindu mythology gods are supposed to reside at the north-pole, where one civil day is the same as one sidereal solar year for the other places; also the demons are supposed to reside at the south pole so that their civil day also is equal to a sidereal solar year. But, what is day to the gods is night to the demons and vice-versa.

If we call the earth, the moon of the moon so to say, which it is so, when it is the moment of new moon for the earth, it is the moment of Full Moon to the Moon. Thus what is a Chāndra-māsa to the earth may very well be called with respect to the Moon a Bhauma-māsa. In Hindu mythology the manes are supposed to take residence on the surface of the Moon. We know from modern astronomy that the moon revolves about her axis once roughly in a lunation. Thus for 'the man on the moon', a day is roughly equal to our lunation. So if the manes were to reside on the moon, their day is equal roughly to a lunation of ours. We say '*roughly*' because as we see under the chapter of the lunar eclipse, the moon *almost* shows the same face to the earth on account of what are called 'librations in longitude'. On this count the time of rotation of the moon does not exactly coincide with the time of a lunation.

A civil day for a place is also the civil day for every place of the earth so that it is called the earth's day. By 'day' here we do not mean the time when the Sun is above the horizon, for that time differs from place to place on the earth. A civil day is the sum-total of the duration of day and the duration of night for any place and it will be seen that this is the same for the entire earth except at the places having perpetual day. The word Saura-Sāvana is used to signify that the time that elapses between two consecutive rises of any other planet is termed the Sāvana

day pertaining to that planet. Thus the Chāndra Sāvana day is roughly equal to 24 hrs–52[1] and this is the longest of the Sāvana days pertaining to the planets. The shortest Sāvana days is that of the Saturn, since Saturn moves a very little along the ecliptic and his Sāvana day is therefore just a little longer than the sidereal day. This nomenclature brings in the phenomenon that the Sāvana day of a retrograde planet happens to be less than a sidereal day. But in a given time, which is sufficiently long like a yuga, the Sāvana days of a planet are equal to the number of sidereal revolutions in that period minus the number of the planetary revolutions in the same period. This will be constant for a given planet in such a long period, though individual days happen to be some shorter and some longer than a sidereal day.

The modern sidereal day is the time between two consecutive rises of the equinoctial points and as the equinoctial points have a slow retrograde motion, the modern sidereal day is just a little shorter than the Hindu sidereal day, which does not take cognizance of the revolution of the equinoctial points round the earth in reckoning diurnal motion. The Hindu astronomers speak of the revolution of the stars only around the earth in the context of diurnal motion. It will be noted that the Saura-Savana day or the civil day will not be of the same duration, since the Sun has unequal motion amongst the stars from day to day. When the Sun is in perigee and has the max. daily velocity, his Sāvana day at that moment is the longest and the Saura-Sāvana day when the Sun is in apogee will be the shortest.

Though in Indian Chronology a day is divided into sixty ghatis for convenience, these ghatis are evidently longer than the Nākshatra-ghatis or the sidereal ghatis, which are of a fixed duration. Thus a Sāvana ghati is a little longer than a Nākshatra-ghati and what is more, a Saura-Sāvana ghati is of a variable duration from day to day, the variation being of course very small.

The concept of a month originally arose out of the phenomenon of new-moons, for, this phenomenon alone appeals to every lay man, when he could not sight the Moon. The concept of an year arose originally out of what is called a tropical year, which is the time between two consecutive conjunctions of the Sun with an equinoctial point and which it is that makes the seasons recur. Thus the primitive man must have had the concept of an year when he once saw the mango trees blossoming and again when he saw them blossom. This is why the Hindus celebrate the new year's day with eating the neem flower with unripe mangos, which goes by the phrase Nimba-Kusuma bhakshaṇam or eating the neem flower. In the Vedic times, however, the year began with the luni-Solar month called Mārga-Sīrṣa, which brings in the new year crops. In the Vedic sacrifices, there is thus what is called the Āgrahāyaṇeṣti, where the word Āgra-hāyaṇa means the Mārga-Sirṣa month. The etymology of the word is that अग्रे हायनं यस्य तत् आग्रहायणम् ie the year is ahead of this month, which therefore is the beginning month of the year. This is also why the ancient lexicon named Amarakos'a enumerates the months from Mārga-Sīrṣa. This is the month when the full moon occurs when the Moon is in the star Mṛgas'ira. We have also an inkling from this that the vernal equinoctial point was probably situated in the star Mṛgas'ira. The Veda however enumerates the stars from the star Krittică, and we have a statement in the S'ata-patha-Brāhmaṇa that "एता ह वै कृत्तिकाः प्राच्यै दिशो न श्च्यवन्ते ie Behold! these stars which go by the name the Krit-ticas do not deflect from the east-point. As this group of Krittikas is situated on the ecliptic, the statement that they were rising in the east signifies that the Vernal equinoctial point was situated in the Kritticas". Arguing about the situation of the vernal equinoctial point in the so-called Vedic times, the late Lokamānya Bāla Gangā-dhara Tilak concluded that Vedic literature must have had its beginning about eight thousand years ago.

From the original concepts of the month and the year, further concepts of the different kinds of month and the year arose with the advance of astronomical knowledge. We shall deal with these different kinds of the month and the year in their respective contexts.

In these two verses, we have the definitions of Saura-māna, Daiva-māna, chāndra-māna, Paitra-māna, Sāvana-māna and Nākṣatra-māna, six of the nine mānas, ie measures of time.

Verses 21, 22, 23, 24, 25. The four yuga-pādas named Krita, Tretā, Dwāpara and Kali consist of 4 × 432000, 3 × 432000, 2 × 432000 and 432000 mean solar years respectively, the sum total of which consisting of (4+3+2+1) × 43200 = 43,20000 mean solar years, is calleda yuga. Each of the yuga-pādas above are inclusive of what are called their respective Sandhyās and Sandhyāmsās which constitute one-twelfth of their own durations.

A Manu's duration consists of 71 yugas and 14 Manus' duration is reported to be the day-time of Brahma, whose night is also of an equal duration.

The duration of a Manu, known as a Manvantara has a Sandhyā-Kāla on either side, ie before and after, equal to one Krita. If these are taken into account, the day-time of Brahma amounts to one thousand yugas and it goes by the name a Kalpa so that a complete day of Brahma equals two Kalpas. The life-duration of Brahma consists of one hundred years on this scale (where one year = 360 days). This life-duration of Brahma goes by the name Mahā-Kalpa, as reported by elders. In as much as Time was without a beginning and will have no end either, I do not know how many Brahmas have gone before.

Comm. In these verses we are given what is known as Brāhma-māna, the seventh of the nine mānas. Incidentally we are also given the measures yuga-pādas,

Manvantaras and a Kalpa. Since in many ancient astronomical works, the revolutions of the planets and the planetary points like nodes, apogees or aphelia are given as integers during the course of a yuga, the concept of a yuga must have arisen as follows. The durations of the sidereal revolutions of the planets and the apogee of the Moon and its node having been ascertained by observation, a period was calculated in which are contained integral multiples of those durations. In other words a yuga of 4320000 mean solar years is construed as the period in which the planets the node and apogee of the Moon make an integral number of revolutions with respect to the stars.

We have excluded here the aphelia and the nodes of the planets, as we shall see later that their sidereal revolutions were not based on observation but by an assumption that those points also must be having an integral number of revolutions during the course of a Kalpa, having started at the beginning of the Hindu Zodiac ie the beginning point of Aswini at the beginning of the Kalpa. On this assumption cited, and using indeterminate analysis the number of their sidereal revolutions were got as reported by Bhāskarācharya in his Commentary in the chapter Bhaganādhyāya. He has given us a clue that the numbers of sidereal revolutions of the planets including the node and apogee of the Moon were originally determined by observations though he appeals to Āgama that those numbers were given by Āgama, as accepted and transmitted by Brahmaguptācharya. Even in the Upapatthis or proofs that Bhāskarācharya gives regarding the numbers of sidereal revolutions known as Bhagaṇas, we perceive that he consciously commits the logical flaw known as इतरेतराश्रयदोष as we are going to show in that context. The proofs he adduces are indeed based upon Aryabhatācharya's verse 48. Goḷapāda namely

क्षितिरवियोगाद्दिनकृत्, रवीन्दुयोगात्प्रसाधितश्चेन्दुः
शशिताराग्रहयोगात् तथैव ताराग्रहाः सर्वे

and on Brahmaguptācharya's verse 12, ch. 20 namely

ज्ञातं कृत्वा मध्यं भूयोऽन्यदिने तदन्तरं भुक्तिः
त्रैराशिकेन भुक्त्या कल्पग्रहमण्डलानयनम्

Aryabhatācharya and his immediate followers Lallācharya and Vateswarācharya make the yuga-pādas of equal duration. Brahmaguptācharya criticises Aryabhata for having said so against the Canons of the Smṛtis as well as Romaka for having ignored the concept of yugas, manvantaras and Kalpa, as this he deems as a heresy (Vide verses 9 & 13 ch. I).

Indeed there seem to be two schools among the ancient Hindu Astronomers one of Brahmagupta who was followed by Sripati, Bhaskara and a number of others and the other of Aryabhata who was followed by Lalla, Vateswara, Bhaskara I, and a host of others mostly hailing from Kerala.

There is an Aryabhata who has been termed Aryabhata II and who was the author of a book named Brihad-Aryabhatiyam or Mahā-Siddhanta as it is also called. M. M. Sudhakara Dwivedi mentions in his Ganaka-Tarangaṇi, that this Aryabhata should have existed after the author of Modern Surya Siddhanta. Aryabhata I, Aryabhata II, many of the Kerala astronomers used a different nomenclature to signify numbers, denoting them by letters. Thus one of the distinguishing features of the Kerala school of astronomers (not all of them) seems to be to use letters for numbers.

One Kalpa = 14 Manvantaras = 14 × 71 yugas + 15 Sandhis in between the Manvantaras each equal to a Krita ie 4 Kalis = 994 yugas + 60 Kalis = 994 + 6 yugas = 1000 yugas = 4320000000 mean Solar years.

Bhaskara speaks of Sandhyas and Sandhyamsas each of them equal to $\frac{1}{12}$ of the yugapādas.

Thus one Kali = 432000 years = 1200 Divyābdas (gods' years each year being equal to 360 Solar years) = 1000 +

100 + 100 Divyābdas since $\frac{1}{12}$ × 1200 = 100 = measure of the Sandhyā and Sandhyamsa times. At this rate Dwapara = 2000 + 200 + 200 Divyābdas and so on. However, the measure of the Sandhyas with respect to a Manvantara does not follow this one-twelfth rule, because a Krita is not $\frac{1}{12}$th of the Manvantara. Thus 1000 Divyābdas = one Kali excluding Sandhyā and Sandhyamsa, whereas 1000 yugas = one Kalpa including the Sandhyas and Sandhyamsas.

When a yuga was conceived as a period wherein the planets make an integral number of revolutions, it goes without saying that they make integral numbers of revolutions in a Manvantara or a Kalpa. When a Kalpa was conceived as the period in which the slow-moving planetary points aphelia and nodes also make an integral number of revolutions, one wonders how a manvantara was conceived. It is further peculiar why such an odd number 71 was chosen, when it was said that 71 yugas make a Manvantara One also wonders why the modern Suryasiddhānta says that after the Kalpa began, creation started only after 47400 Divyābdas whereas neither Brahmagupta nor Bhāskara speaks of this. As a matter of fact Bhāskara mentions later the number of years that had elapsed upto the beginning of the Saka era, as equal to 1972947179, but does not speak when the creation of planets and stars began actually.

Verse 26. Half the life-period of the present Brahma has elapsed; some said that only eight and half years of his life has elapsed — Let the Āgama or tradition be whatsoever; we don't have any need of knowing it because the planetary positions have to be computed only from the beginning of this Kalpa.

Comm. Vateswara it was that mentioned that only eight and half years of the present Brahma had elapsed (Vide verse 10 Madhyādhikara ch. I Vateswara Siddhānta) Vateswara prescribes that Ahargana or the collection of

days has to be calculated from the birth-time of this Brahma, but Bhāskara rightly points out that it is a waste of labour, for, all the planets must have returned to the Zero-point of the zodiac ie the beginning of the Star Aswini at the beginning of this Kalpa and hence it is sufficient to calculate only from the beginning of this Kalpa. Further Bhāskara states that when the very planets did not exist during the last elapsed night of Brahma, what is the fun of calculating their positions.

There is also a tradition that there are nine Brahmas and that the present one is the very first. This tradition Bhāskara does not mention, because he exclaims that he does not know how many Brahmas have gone by, Time being without a beginning.

Verse 27. In as much as the creation started only from the beginning of this Kalpa which is the present day-time of Brahma, and because deluge takes place at the end of the day-time, the question of Computing the planetary positions arises only when the planets exist. If some (the allusion is to Vateswara) propose to compute the planetary positions even when the very planets did not exist, may we salute those great people !

Comm. Not necessary.

Verse 28. Six Manus have elapsed in this Kalpa, thereafter twentyseven yugas, as well as three yugapādas namely Krita, Tretā and Dwāpara. Further 3179 years of this fourth yugapāda namely Kali have elapsed by the end of the Saka king (which moment was the beginning of the Saka era). Hence in the present Kalpa ie the day-time of this Brahma, 19729 47179 years had elapsed upto the beginning of the Saka era.

Comm. The computation is as follows:

6 Manvantaras = 6 × 71 × 10 Kaliyugas since each Manvantara Consists of 71 yugas and a yuga Consists of

ten Kaliyugas (one yuga = Krita + Tretā + Dwāpara + Kali = 4 + 3 + 2 + 1 = 10 Kaliyugas). The Sandhis that were there in between the Manus and in the beginning of the first Manu are seven and each Sandhi being equal to one Krita or four Kalis, the seven Sandhis = 7 × 4 = 28 Kaliyugas.

Further it is stated that 27 yugas had elapsed in the present seventh Manvantara known as Vaivasvata which are equal to 27 × 10 = 270 Kaliyugas. Further in the present yuga, Krita, Tretā and Dwāpara had elapsed equal to 4 + 3 + 2 = 9 Kaliyugas.

Thereafter in the present Kaliyuga 3179 years elapsed upto the beginning of the Saka era. Thus totalling we have 4260 + 28 + 270 + 9 kalis + 3179 years.

= 4567 kalis + 3179 years.

= 4567 × 432000 + 3179 years.

= 179 294 7179 year as mentioned.

Verse 29. The six Manus that went before the present Vaivasvata were Swāyambhuva, Swārociṣa, Auttama Tāmasa, Raivata and Cakshusha.

Comm. Clear—Upto this point we have seen the Brāhmamāna, the seventh of the nine mānas.

• *Verse* 30. The Sāmhitikas declare that a Samvatsara is equal to the ṭime of a mean sidereal revolution of Guru the Jupiter (This is the Bārhaspatyamāna). The ninth māna named the Manushyamāna is composite of the four mānas (as detailed in verse 31).

Comm. The years of the Bārhaspatya māna are enumerated as Vijaya, Jaya etc which are sixty in number. The same names are also adopted in the cāndramāna ie the luni-Solar reckoning (In vogue in the Andhra Pradesh and some other provinces too) only with the difference that prabhava is taken as the starting year in which sequence Vijaya happens to be the twenty-seventh year. Since the

mean sidereal revolution takes roughly 11.8 years, five revolutions take 59 mean solar years. The sixtieth year thereafter of the Jovian Cycle is considered as an Adhi-Samvatsara thereof so that the luni-Solar reckoning as well as the jovian reckoning get wedded together. Thus to get the jovian year we have simply to add 27 to the luni-solar year. The wedding of the two reckonings has its analogy in the process of intercalation which weds the solar reckoning with the luni-Solar only with the difference that months of the latter reckoning are set apart as Adhika or extra months. Just as the process of intercalation brings in its train what is called a Kshayamāsa as per the convention that, that month would be set apart as an Adhikamāsa, which does not carry a Samkrānti ie entrance of the Sun into the next Rasi, in which process there appears a month in which there may occur two Samkrāntis which is hence considered as a Kshayamāsa, just in a similar way that luni-Solar year in which the Jupiter enters the next Rasi, is supposed to be normal whereas that year in which such an entrance does not take place is deemed an Ādhika year and set apart while that year which carries two entrances is deemed as a Kshya year. The process of intercalation which weds together the solar and the luni-solar reckonings will be elucidated further in its appropriate context. The word Sāmhitikas means the authors of the works called Samhitas like the Varāha-Brihatsamhita etc.

Verse 31. The manushya-māna or that which men follow, adopts the year, the Ayana the Ṛtu or the season (six in number during an year) and the yuga according to the movement of the Sun ie according to Sauramāna the months and the thithis according to the luni-solar reckoning ie according to Chāndramāna the Vratas, upavāsas, treatment of diseases, deliveries of ladies, the names of the weeks all these according to the reckoning of civil days ie according to Sāvanamāna and finally the ghatis

to the

Comm. A lunation is divided into thirty tithis, which go by the names pratipat etc. of which there are fifteen in the brighter half of the lunation and fifteen in the dark fortnight. The pratipat tithi is that duration of time beginning from the moment of New Moon and ending when the Moon has over taken the Sun by 12° ; the second tithi named dwitiyā begins at the end of pratipat and lasts upto the point of time when the moon's elongation is 24° and so on. Thus the tithis are seen to be of unequal length in as much as both the Sun and the Moon have unequal motion. The mean duration of a tithi is seen to be a little less than a civil day since a lunation has ronghly 29½ civil days. There is a convention that the tithi which is current at a Sun-rise will be considered to be the tithi of the whole day. As per this convention it so happens that a tithi lasts just a little after Sun-rise and the next tithi vanishes during the same day so that the next but one will be taken for the next day. This vanishing tithi is known as a Kshaya tithi which is also called a Kshayāba. Again it so happens that a tithi may be current at two consecutive Sun-rises beginning a little before the Sun-rise of the first day and extending a little after the next Sun-rise. Such a tithi is called Dina-Traya or a tithi which touches three days. In this matter if seems as though we have gained a tithi but ultinately as a tithi must be less than a civil day, it so happens that on the average there will be a Kshayāha ronghly in 64 tithis. We shall see more about this matter subsequently.

Verse 32. Thus there are nine mānas Mānava, Divya (or of gods), Bārhaspatya (Jovian) paitra, Nākshatra, Saura, Chāndra, Sāvana and Brāhma. But the planetary positions are to be computed by men by their own māna.

Comm. Not necessary.

Here ends the Adhyāya known as Kāla-māna in the **Madhyādhikāra.**

MADHYĀDHIKĀRA – SECTION II
BHAGAṆĀDHYĀYA

Verses 1 to 6. The number of sidereal revolutions of the Sun during a Kalpa is 4320000000. It is also the number of those of Mercury and Venus, and those of the S'ighrocchas of the planets Mars, Jupiter and Saturn.

The Moon makes 57753300000 sidereal revolutions in a Kalpa, the Mars 2296828522, the Mercury's s'ighroccha 17936998984, the Jupiter 364226455, the s'ighroccha of Venus 7022389492 and the Saturn 146567298. The sidereal revolutions of the apogees of the Sun and the Moon and those of the aphelia of Mars, Mercury, Jupiter, Venus and Saturn in a Kalpa are respectively 480, 488105858, 292, 332, 855, 653, 41.

The retrograde sidereal revolutions of the nodes of the orbits of Moon, Mars, Mercury, Jupiter, Venus and Saturn are respectively 232311168, 267, 521, 63, 893, 584.

Comm. Since Mercury and Venus will be oscillating about the Sun in their apparent motion as seen from the earth in a long period of time, the number of sidereal revolutions made by the Sun is also equal to that made by Mercury and Venus. Since, as we see in the Course of the Spaṣṭādhikāra the Sun plays the part of what is called the S'ighroccha of the three major planets named Mars, Jupiter and Saturn, the number of the Sun's sidereal revolutions is also the same as that of their s'ighrocchas.

The reason why the sidereal revolutions of what are called the s'ighrocchas of Mercury and Venus are the same as the heliocentric sidereal revolutions of Mercury and Venus will be clarified in the spaṣṭādhikāra. The reason why the sidereal revolutions of the major planets

the same as their heliocentric ones will also be clarified in the same adhikāra.

The reason why we have termed the Mandocchas as apogees in the case of the Sun and the Moon and as aphelia with respect to the other planets is that the Sun moves round the earth relatively while the Moon directly moves round the earth whereas the remainning planets move round the Sun while the Sun moves relatively round the earth. In other words the Sun and Moon have apogees whereas the remaining planets aphelia.

The nodes of the planets are the points of intersection of their orbits with the ecliptic which is the apparent orbit of the Sun. In other words the planetary orbits are inclined to the ecliptic which means that their orbital planes do not coincide with the ecliptic plane.

In Hindu Astronomy the Sun, the Moon and also the two nodes of the lunar orbit which go by the names Rahu and Ketu are also termed as grahas along with the other five which are known as Tāra-grahas or planets resembling stars. The etymology of the word 'planet' is that it moves amongst stars; in this respect the nine Hindu grahas also moving among the stars are eligible to have the same appellation though it is not permitted in modern astronomy. But the word graha has a different connotation etymologically namely गृह्णातीति वा गृह्यते अनेनेति वा ग्रहः ie that which seizes the fates of men is known as a graha. In the course of this work we use the word graha and planet synonymously so that we deem the Sun, the Moon and the nodes of the lunar orbit also as planets.

When the ancient Hindu astronomers knew by observation that the nodes of the lunar orbit have a retrograde motion, and could also measure their mean motion, they extended the analogy to the nodes of the other planets also, whose motion could not be measured during anybody's life-

time-Hence the estimate of their mean motion by the Hindu Astronomers naturally went wrong.

Noticing that the Mandoccha of the Moon has a progressive motion which the Hindu astronomers could measure correctly, they extended the analogy to the apogee of the Sun and the aphelia of the other planets whose motion also being very slow could not be measured during the life-time of a man. So here also the estimate of their mean motions of the Sun's apogee and the planetary aphelia went wrong.

Bhaskarācharya has given proofs as to how the sidereal revolutions could be got but as we shall see later in the Spaṣṭādhikāra, his proof occasionally suffers from what is called 'इतरेतराश्रयदोष' ie 'begging the question'. We shall however construct our own proofs at that place deferring them for the present, for, the proofs require an elucidation which obtains in the Spaṣṭadhikāra alone.

Verse 7. The number of diurnal revolutions of the stars in a Kalpa is 1582236450000.

Comm. In fact this number is that of the diurnal rotations of the earth which is equal to the number of apparent diurnal rotations of the stars, if the earth is deemed as fixed. The only Hindu Astronomer who made bold to say that the earth is rotating and not the stars came under criticism by Brahmagupta and the latter Hindu astronomers. Aryabhata said अनुलोमगति: नौस्थ: पश्यत्यचलं विलोमगं यद्वत्, अचलानि भानि तद्वत्. समपश्चिमगानि लङ्कायाम्" ie Even as a man stationed on a moving boat perceives that the trees etc on the banks of the canal, river or lake to be moving in the opposite direction supposing himself stationary, so also men stationed on the surface of the earth (which is like a moving boat) perceive the actually stationary stars to be moving directly from east to west at Lanka ie the equator".

Why none of the latter Hindu astronomers, though many of them could intuit this simple phenomenon, boldly came out asserting this, is rather mysterious. Even today there is such an irrational orthodox type of scholars who are not in touch with modern astronomy, holding the view that the earth does not move.

In Hindu Astronomy it was postulated that there is what is called the pravaha wind, which effects moving of the entire stellar universe along with the planets from east to West. It is a very simple matter to visualize earths' rotation, instead of supposing that the entire stellar Universe is being driven round the earth. The absurdity in this latter supposition might not have been clear to the orthodox type of the Hindu Astronomers, for, they could not measure the dimensions of the giant stars and super-giants, which are everyone of them mighty Suns. Though, however, the dimensions of the Sun were known to them to be far greater than those of the earth, it did not occur to them why a mighty Sun should go round a pigmy earth. Or even if it occured to astronomers like Bhaskara, they dared not to go against the puraṇic tradition.

Since the stars do not move among themselves while partaking this diurnal motion, the entire starry skies are obliged to go round the earth as a rigid structure, if we suppose that the earth is not rotating about, herself. This kind of supposition is just like a revolving person, revolving about himself and claiming that the entire Universe is revolving round him and not he about himself.

Bhaskara gives the proof of getting this number of diurnal rotations of the stars during an year in the verses 5–7 of Madhyagati Vāsanā, समं भसूर्यावुदितौ etc. as follows. Suppose a star and the Sun rise together today. Tomorrow the star will have arisen earlier than the Sun who will have moved towards the east of the star by his own (apparent) daily motion. So tomorrow's sun-rise will get

belated by the duration of time that the arc of the ecliptic covered by the Sun's today's motion takes to rise. This duration of time is variable on two counts; first by the variable motion of the Sun and second by the obliquity of the ecliptic on account of which even equal arcs of the ecliptic will not rise in equal times. In other words the duration of time between two consecutive Sun-rises will not be the same. This duration of a particular day can roughly be calculated by the rule of three as follows. Let the Sun be in a particular Rasi, the rising time T of which could be computed; let the Sun cover an arc of x° in that Rasi on that day. Then the time taken by that arc to rise is $\frac{xT}{30}$, where a Rasi consists of 30°. This time computed in Sidereal measure added to 60 Sideral ghatis is equal to the length of the day. During the course of an year ie the time taken by the Sun to move round the ecliptic starting from the Zero-point of the zodiac and again returning to the same point, the Sun will have made one revolution less than the stars. Thus if 'R' the number of diurnal revolutions of the Sun (where R will be not an integer) during an year or what is the same the number of Sīvanā or civil days during an year, they are equal to R + 1 sideral days. Hence the number of sideral days in a kalpa will be equal to the number of civil days in a kalpa together with the number 4320000.000 which is the number of revolutions made by the Sun relative to the stars.

In this context we are to know the number of civil days in a kalpa. The proof given by Bhaskara in Gaṇitadhyāya under verses 1–6 in Bhagaṇopapatthi is as follows. Draw a circle on a horizontal plane and place a vertical pole called gnomon at the centre of the circle. Observe the point of intersection of the gnomon's shadow with the circumference of the circle at Sun-rise on a day in the Uttarāyaṇa ie during the course of the Sun's north-word journey, just at the time when his rising point is very near east point and also to the south thereof. Then from

that day go on counting the number of days, which will be 365 when the Sun again rises very nearly at the same point and just to the south of the east point. It will be found that the Sun will rise the next day just a little to the north of the east point, which means that the Sun has taken 365 days and a fractional part of a day to complete his revolution round the stars. Having noted the two points of intersection of the gnomonic shadow with the circle, on those two consecutive days when the Sun happens to rise just a little to the south and then on the next day just a little north of the east point, and having measured the arcs in minutes between those points of intersection and the western point of the horizontal circle (western because the gnomonic shadow of the rising Sun is cast towards west) then the following rule of three is to be applied. If during 60 ghatis of the day, the sum of the arcs in minutes is covered, what will be the time taken by the shadow to traverse the arc between the west point and the northern point of intersection. This added to 365, gives the number of civil days in an year. Here it will be noted that this year is tropical because rising in the east signifies the Sun's position at an equinox.

Verse 8. The number of solar days in a kalpa is equal to 1555200000000 and of the lunar days or tithis is 1602999000000.

Comm. The solar days here cited are counted at the rate of 360 per year; and the lunar days at the rate of 30 per lunation. Tithi is defined as mentioned by us under verse 31 of the previous section. Since there are thirty tithis in a lunation, their enumeration is quite alright; but there seems to be a little oddity in saying that there are 360 solar days in an year. This kind of a solar day is a little longer than a civil day and does not correspond to any particular motion of the Sun, say for example the time taken by the Sun to move a degree along the ecliptic. definition of solar days pertains only to a stipulation

360 solar days constitute a solar year and no tion is given for a single solar day. This definition of solar days, though apparently artificial, has some significance, namely that the difference of the solar and lunar days defined above constitutes 1593300000 Adhikamāsas in a kalpa at the rate of 30 tithis per month. In other words the difference between the solar and lunar days, is the number of tithis that the luni-solar reckoning gains over the solar. This topic will be dealt with later.

Verse 9. The number of civil days in a kalpa is equal to 1577916450000; the number of the diurnal revolutions of the stars minus the number of sidereal revolutions of any particular planet constitute the days of that particular planet with respect to the earth.

Comm. The civil days in a kalpa are evidently the number of Sun-rises. These are as mentioned before the difference of the number of diurnal revolutions of the stars and the number of the sidereal revolutions of the Sun. By analogy, the number of the days of a particular planet with respect to the earth or what is the same the number of risings of that planet in a kalpa as seen from the earth, is the difference of the number of the diurnal revolutions of the stars and the number of the sidereal revolutions of that planet in a kalpa. Thus we have Saura-Ku-dināni, Chāndra-Ku-dināni, Bhauma-Ku-dināni etc, where the word 'Ku' means the earth. The Saura-Ku-dināni are the civil days defined before. It will be noted that the Chāndra-Ku-dināni are not the lunar days.

Verse 10. The number of Adhikamāsās or intercalary months in a kalpa is equal to 1593300000 and the number of Dina-Kshayas is 25082550000.

Comm. A luni-solar year as per Cāndramāna ie the luni-solar reckoning consists of twelve lunations; as such its length falls short of that of the solar year by 11 days 3 ghatis, 52 Vighatis and 30 Sūkshmaghatis. If both the

solar and luni-solar years begin simultaneously this year, by the end of the luni-solar year, it will have gained over the solar year the above-mentioned days. This difference which goes by the name Adhimasa-Sesha or Suddhi accrues to the length of a lunation in about 32½ solar months. Unless this accruing difference is set apart by some device, and the beginnings of the two years again brought together, the luni–Solar year looses its significance of an year, for, it does not accord with seasons. It being a convention that an year should begin with the spring, if the luni–Solar year also is to begin with the spring, it is to be wedded to the solar year by some device. The device adopted in this behalf was to leave out a month in the luni-Solar reckoning as soon as it could be seen that a month has been gained by this reckoning over the solar. This knowledge is had from the following fact. The zodiac is divided into twelve equal portions called Rasi's beginning from the first point of the Hindu Zodiac, ie. from the first point of the asterism division called Aswini. The Sun traverses each of these Rasis in one Solar month, and on account of the unequal motion of the Sun, these solar months are of unequal length. The entrance of the Sun from one Rasi into another is called a Samkrānti, and the moments of Samkrāntis are held to be holy for religious purposes. The solar month being a little longer than a lunation, normally a Samkrānti occurs in a lunation. But it so happens that in a particular lunar month this Samkrānti might not occur. The Suddhi which is the time between the moment of New Moon and the subsequent Samkrānti and which is therefore the time gained by the luni–Solar reckoning over the Solar, having accrued to a lunation, it is an indication that the luni–Solar reckoning has gained a lunation over the Solar. That lunation not carrying a Samkrānti is termed an Adhikamasa and is left out. That it is left out is connoted by the word Adhika which means extra, as well as by the convention that no auspicious celebrations like a marriage etc. should not take

during that month. The next lunar month is termed as the Nija-māsa or the month which is the true. In this convention however, there may occur two Samkrāntis in one particular lunar month. For this to happen two criteria are to be satisfied namely that, that particular lunar month must be greater than the concurrent solar month and secondly one of the two Samkrāntis must occur immediately after a New-moon so as to permit the second Samkrānti to occur just before the lapse of the lunar month. The Sun coming to his perigee roughly about the lunar month Mārgasira, and as such having the quickest motion at that point, he covers the length of the Rasi of 30° within the shortest span of time making that Solar month shorter than the corresponding lunar month. Since the months Kārtica and pausha, one before and the other after Mārgasira, also being thus longer than the corresponding solar months, two Sankrāntis could occur, if at all, in these three lunar months. If they occur, that lunar month is termed a Kshayamāsa. Since two lunar months are to correspond to two Samkrāntis, convention has it that two lunar months lapse .simultaneously' during that lunation, the previous lunation running during the forenoon and the subsequent lunation during the afternoon. Thus a Kshaya māsa is also called a yugalibhūta māsa or a twin māsa, which therefore makes one lunation virtually vanish. On account of this vanishing', it is called a Kshayamāsa. After what an interval of time this Kshaya māsa occurs how a month precedes it having no Samkrānti and how again a lunation follows it without a Samkrānti will be seen in a subsequent context. This convention pertaining to the institution of an Adhikamāsa goes by the name 'Inter-calation'.

It was mentioned before that on an average an Adhikamāsa occurs once in 32½ solar months roughly-Also, Adhikamāsas are the excess of lunations over the solar months in a given period. There being 51840000000 solar months in a Kalpa and 53433300000 lunations the number of Adhikamāsas is therefore 1593300000 as mentioned.

Kshayāhas were explained before and their number in a Kalpa is the difference of the civil days and the tithis.

Verses 11, 12. The numbr of solar months in a Kalpa is 51840000000; the number of lunar months is 53433300000. The number of solar months heing subtracted from the number of lunations, we have the number of Adhikamāsas—The number of solar days together with the days of Adhika months are equal to the lunar days or the tithis; or again the lunar days minus the Kshayāhas are equal to the number of civil days or the reverse will be had by a reverse process.

Comm. Already explained.

Verse 13. The excess of the sidereal revolutions of the moon over the number of the Sun's sidereal revolutions is equal to the number of chāndramāsas or lunations.

Or again the excess of the sum of the sidereal revolutions of the moon and the tithis over the sum of the lunations and the diurnal revolutions of the stars is equal to the number of Kshayāhas.

Comm. The first part is clear, Regarding the second, let the number of lunations be x and the number of the sidereal revolutions of the moon be y. Then $y-x=z$, the number of the sidereal revolutions of the Sun because $y-z=x$ from the first part above. If now, the number of the diurnal revolutions of the stars be t, then $t-z=t-(y-x)=t+x-y=$ civil days. Subtracting these civil days from 'U' the number of Tithis, we have the number of Kshayāhas namely $U-(t+x-y)=(U+y)-(t+x)$ which accords with the statement.

Verse 14. The number of Adhikamāsas is equal to the excess of the number of sidereal revolutions of the Moon over thirteen times the number of sidereal revolutions of the Sun.

Comm. Let x and y be the numbers of sidereal revolutions of the Moon and the Sun respectively. The $x-y$ is evidently the number of lunations as mentioned before. Also 12 y is the number of solar months which if subtracted from the number of lunations will give the number of Adhimāsas ie $x - y - 12y = x - 13y$ = Adhikamāsas as mentioned.

This completes the Bhaganādhyāya of Madhyādhikāra.

GRAHĀNAYANĀDHYĀYA ग्रहानयनाध्यायः

Verse 1. To Compute the Ahargaṇa the collection of days from the beginning of Kalpa ie from the beginning of creation. Multiply the number of sidereal solar years from the beginning of Kalpa by 12; add the number of elapsed lunar months; multiply by thirty; add the number of elapsed tithis. Let this number be x. Then $\left[\frac{x \times A}{s}\right]$ the integral number obtained by dividing the product of x and A the number of Adhikamāsas in a Kalpa by s the number of solar days thereof gives the number of elapsed Adhikamāsas. Multiply this number of Adhikamāsas by 30 and add to x. The result gives the number of elapsed tithis, Let this be y. Then $\left[\frac{y \times K}{T}\right]$ the integral number obtained by dividing the product of y and K the number of Kshayāhas in a Kalpa by T the number of Tithis in a Kalpa, gives the number of Kshayāhas. Subtracting this number from y, we have the Ahargaṇa ie the number of the elapsed civil days from the beginning of Kalpa. This Ahargaṇa has its beginning on Sunday and is itself constituted of mean solar days. While computing the Adhikamāsas or Kshayāhas, the integral numbers of the quosients alone should be taken rejecting the remainders.

Comm. The elapsed number of solar years is directed to be multiplied by 12 in the beginning. This number does not constitute purely solar years. Solarity was secured upto the point of the last intercalation of an Adhikamāsa and thereafter one or two luni-solar years would have been added. But Construing these one or two luni-solar years as mean solar years does not make a difference while computing the Adhikamāsas, for the following reason. Adhikamāsas normally occur once in three years. Even

supposing that the number of the elapsed years contain three luni-solar years after the year carrying the last intercalary month, the error in construing them to be solar will be roughly minus one solar month. In other words the difference between three mean solar years and three luni-solar years will be roughly one solar month. Since one Adhikamāsa occurs roughly in $32\frac{1}{2}$ solar months, so the number of Adhikamasas obtained by construing three luni-solar years as mean Solar years will be in default by roughly $\frac{1}{32\frac{1}{2}}$ or $\frac{2}{63}$. As we are counting only the integral number of Adhikamāsas obtained as a quotient rejecting the remainder, the above default of $\frac{2}{63}$ should not normally effect the quotient. Most rarely, however, it might effect the quotient by one, for which provision is made by Bhaskara in verse 3 under Adhimāsādinirṇaya section, Madhyamādhikara namely स्पष्टोऽधिमासः पतितोऽप्यलब्धः etc. which means that if the quotient is in default by one, where it is definitely known that one more Adhikamāsa did occur, we are directed to add one and if we certainly know that the quotient contains one more Adhikamāsa, when the Adhikamāsa is shortly to occur and has not occrued we are directed to substract one from the quotient. This principle of सैकत्वनिरेकत्व is to be observed even in the context of Kshayāhas ie we are directed to add one or subtract one from the number of civil days by adjusting the Ahargaṇa to the week-day on which the Ahargaṇa is sought to be found.

Thus construing the elapsed number of years to be solar, multiplying them by twelve we have the elapsed solar months. Adding to this the number of the elapsed luni-Solar months construing these to be solar, which fact also does not affect normally the number of Adhikamāsas to be obtained (for the same reason above) and multiplying by 30 and adding the elapsed tithis we have the elapsed number of solar days (This number may not be exactly the

elapsed number of solar days, for, we have construed the one, two or three luni-Solar years as mean solar as well as the elapsed lunations of the present year as solar months; but this difference does not affect the computation of Adhikamāsas as mentioned above). Then, as we are given that 15933,00000 Adhikamāsas occur in 1555200000000 solar days of the Kalpa, if x be the number of solar days found

give the number of the

Adhikamāsas. Adding these Adhikamāsas multiplied by 30, to x the solar days obtained above, we have the Tithis elapsed upto the day in question. If now, we subtract the Kshayāhās from these Tithis, we shall have the number of Sāvanāhas or the Ahargaṇa required. Since in 1602999000000 Tithis of the Kalpa there will be 25082550000 Kshayāhas, if y be the Tithis above obtained, $\frac{y \times 25082550000}{1602999000000}$ will be the Kshayāhās from the beginning of the Kalpa upto the day in question; subtracting these from y, we have the required Ahargaṇa.

Verse 4. Computation of the planetary positions.

The Ahargaṇa multiplied by the number of sidereal revolutions of a planet and divided by the number of civil days in a Kalpa gives the planet ie its number of revolutions upto the day concerned both integral and fractional.

Comm. Applying 'Rule of three', if in C the number of civil days in the Kalpa, the planet makes P sidereal revolutions how many revolutions would have been made in A the Ahargaṇa? The answer is $\frac{A \times P}{C}$. In this, the integral quotient gives the number of complete revolutions made; the remainder, multiplied by 12 and divided by C again, gives the number of Rasis covered by the planet from the Zero-point of the Zodiac, and again the remainder by 30 and divided by C gives the number of deg-

rees covered in the next Rasi; proceeding thus, the planetary position could be had next in minutes and then in seconds of arc also. This position is the mean position of the planet, at the time when the mean Sun is very nearly at the Eastern horizon at Lanka. Why it is said "very nearly at the horizon" will be clear in the context of udayāntara Samskāra to be dealt with later in Spaṣṭādhikāra. To obtain the planetary position at the moment when the mean Sun is exactly on the horizon, we have to apply what is called the Udayāntara correction and again to obtain the position at the moment of True Sun-rise we have to apply what is called the Bhujāntara correction, both of which will be dealt with in Spaṣṭādhikara. Having thus got the mean planetary position at True Sun-rise, we have to apply one or two as the case may be, corrections to obtain the True planetary position, besides effecting two more corrections known as Desāntara and chara to obtain the True planetary position at the time of True Sun-rise not at Lanka but at the place concerned.

Verse 5. To obtain the position of the mean Moon, when the mean Sun is known from what is called Avama-Seṣa.

The Avama Seṣa divided by 131490000000 in degrees is to be added to twelve times the elapsed tithis and the result added to the Sun's position gives the position of the Moon. Conversely the position of the Sun can be had from the position of the Moon.

Comm. The moon's longitude minus the Sun's longitude known as elongation is called Vyarkēndu, which divided by twelve gives the number of elapsed tithis; a lunation which is the time in which the Moon overtakes the Sun by 360°, contains thirty tithis and a tithi is defined as the time, in which the Moon overtakes the Sun by 12°, beginning from the moment of conjunction ie Amāvāsyā. Hence if the Sun's longitude be x°, aud y be the number of

elapsed tithis, integral or fractional $(x + 12y)^\circ$ will be the longitude of the Moon. If, however, we consider y only as the integral number of the elapsed tithis, we will have obtained the Moon's position from the formula $(x + 12y)^\circ$ at the ending moment of the tithi on the previous day. The time in between this ending moment of the tithi and the Sun-rise of the day concerned is known as Avama-Sesha, since the Avama-days or Kshayāhas are the difference of days between the number of tithis in a given period and the number of civil days during the same period. In other words, the excess of a civil day over a tithi which falls short of it, is the part of a civil day that contributes towards the number of Kshayāhās in a given period. Thus we have to compute the increase in the Moon's longitude during the aforesaid Avama-Sesha to obtain his longitude at the Sun-rise of the day concerned. But this Avama-Sesha is of the form $F\left\{\frac{t \times K}{T}\right\}$ where 't' is the number of elapsed tithis upto the Sun-rise of the day concerned K the number of Kshayāhas in a Kalpa, T is the number of tithis in a Kalpa and $F\left\{\frac{t \times K}{T}\right\}$ signifies the fractional part called Avama-Sesha, the integral quotient having given the number of elapsed Kṣhayāhas. Hence writing $F\left\{\frac{t \times K}{T}\right\}$ as $\frac{R}{T}$ where R is the remainder obtained by dividing $(t \times K)$ by T, the Avama-Sesha is of the form $\frac{R}{T}$. This Avama-Sesha being mean solar. it has to be rendered luni-Solar, which process is known as स्वाङ्गीकरणम्. The rule of three used in this behalf is 'If for C civil days we have T tithis of the Kalpa, what shall we have for $\frac{R''}{T}$? The answer is $\frac{R}{T} \times \frac{T}{C} = \frac{R}{C}$. This then is the balance fractional part of a tithi that is there in between the end of the elapsed tithi and the Sun-rise of the day concerned. This part of a tithi is to be multiplied by

12 to give in degrees the increase of Moon's longitude from the ending moment of the tithi on the previous day. Thus this increase is $\frac{R}{C} \times 12$; but C = 1577916450000

$$\frac{R}{C} \quad 12 = \frac{1577916450000}{12} = \quad 131490000000$$

approximately.

Thus this increase is to be added to (x + 12 y)° where x° is the longitude of the Sun and y the integral part of the elapsed tithis so that the Moon's longitude is

$$12\ y)^{\circ} \qquad \overline{131490000000}^{\cdot}$$

Verses 6, 7. Computation of the positions of the Sun and the Moon from the Adhimāsa-Sesha and Avama-Sesha. The Avama-Sesha divided by 27110000000 is termed an additive constant in minutes of arc to the Sun's position ; the same Avama-Sesha multiplied by 13 and divided by 35 is termed such an additive constant to the position of the Moon ; Construe that the Sun's position is given by as many degrees as there are elapsed tithis after the beginning of Chaitra and that the Moon's position is given by Thirteen times the same. Let these positions of the Sun and the Moon be diminished by a number of degrees equal to what is obtained by dividing the Ādhimāsa-Sesha by the number of lunations in a Kalpa. Then add the respective additive constants to the positions of the Sun and the Moon so obtained. The results will be the positions of the Mean Sun and the Mean Moon.

Comm. Here the data are the Adhimāsa-Sesha and the Avama-Sesha and nothing else and the problem set is to find the Mean positions of the Sun and Moon. The Adhimāsa-Sesha is of the form $\frac{R' \times 30}{S}$ where R′ is the remainder obtained while finding the elapsed Adhimāsas

by dividing the product of the number of lunations in a Kalpa and (12 x + y) 30 + t where x is the number of elapsed years, y the number of elapsed lunations t the number of elapsed tithis in the current lunar month at the time when the Ahargaṇa is being computed and S the number of Solar days in a Kalpa. The form of the Avama-Sesha was formerly stated as $\frac{R}{T}$. The procedure given is as follows. In the first place, we are asked to assume that the number of elapsed solar days is equal to the number of elapsed tithis from the beginning of Chaitra ie the beginning of the luni-Solar year, since we do not know when the Solar year began; so, at the rate of 1° per a Solar day (A Solar day is not a mean solar day, Vide definition given before under verse 8. Bhagaṇādhyāya) the Sun's position is given by as many degrees as there are elapsed tithis; and the Moon's position must be 13 times the same since each tithi means an increase of 12° in the elougation and if x° be the Sun's position, the Moon's position must be 12 x° ahead of the Sun ie 13 x°. Having got thus approximate positions of the Sun and the Moon, we have to make amends for the roughness of the assumption made. In assuming that the longitude of the Sun is equal to the number of elapsed tithis, we have overestimated the longitude since it has to begin from the beginning of the solar year. The time in between the beginnings of the luni-Solar year and the solar year is known as the Adhimāsa-Sesha at the time of the beginning of the solar year, measured in tithis; also, we have committed an error in assuming the Sun's rate to be 1° per tithi. This also is due to Adhimāsa-Sesha subsequent to the beginning of the Solar year upto the day concerned. Thus the entire error committed by assuming the Sun's longitude to be as many degrees as there are elapsed tithis from the beginning of the luni-Solar year. is no other than the Adhimāsa-Sesha at the day concerned, But this Adhimāsa-Sesha is of the form $\frac{R' \times 30}{S}$ as mentioned and is in tithis.

This has to be converted into solar days to give us the number of degrees of the error. The conversion is effected by the rule of three as "If T tithis of a Kalpa constitute S solar days of the Kalpa, what number of solar days corresponds to $\frac{R' \times 30}{S}$?" The answer is $\frac{R' \times 30}{S} \times \frac{S}{T} = \frac{R' \times 30}{T} = \frac{R'}{T/30} = \frac{R'}{L}$ where L is the number of lunations in a Kalpa. This $\frac{R'}{L}$ being the number of solar days corresponding to the Adhimāsa-Sesha upto the day concerned, the corresponding longitude of the Sun namely $\frac{R'^{\circ}}{L}$ must be subtracted from the Sun's longitude. Now the question arises whether we have to multiply this $\frac{R''^{\circ}}{L}$ by 13 to be subtracted from the longitude of the Moon; not necessary, enough to subtract $\frac{R'^{\circ}}{L}$ only, for, the Adhimāsa-sesha extends from the preceeding Amāvāsya only when the Moon's longitude was equal to the Sun's longitude. We have thus got the mean positions of the Sun and the Moon at the ending moment of the tithi on the previous day. To get their mean positions at the Sun-rise of the concerned day, we have to make amends for the time in between, which is no other than Avama-Sesha as a fraction of a mean solar day. Here Bhaskara makes an ingenious approximation. The maximum Avama-Sesha could be a tithi only and the Sun moves roughly by his daily mean motion during a tithi. So, the rule of three adopted is 'If for one tithi, the Sun's daily motion is to be reckoned, what for the Avama-Sesha? The answer is Avama-Sesha multiplied by the Sun's daily motion. The Avama-Sesha being of the form $\frac{R}{T}$, $\frac{R}{T} \times m' = \frac{R}{T} \times 59\frac{8'}{60}$

$$\frac{\frac{R'}{T}}{59\frac{2}{15}} = \frac{R'}{\frac{1602999000000 \times 15}{887}} = \frac{R'}{27110000000}$$ very approximately.

This must be added to the position of the Sun to get his position at the Sun-rise concerned. In the case of the Moon the above additive constant of the Sun, is to be multiplied by $13\frac{13}{35}$ because the Moon's daily motion is so many times that of the Sun. Hence the additive constant in the case of the Moon is $x \times 13\frac{13}{35} = 13\,x\,(1 + \frac{1}{35})$ where x is the additive constant in the case of the Sun. This agrees with what Bhaskara has stated.

Verses 8, 9. Another way of computing the planetary positions.

The mean position of the Sun in Rasis minus $\frac{A \times G}{131493037500}$ Rasis where A stands for the Ahargaṇa, G stands for the Sāvana days of the planet concerned in a Kalpa gives the position of the planet in Rasis. Let pandits find out other similar methods.

Comm. The Sāvana days of a planet is the excess of the diurnal rotations of the stars over the sidereal revolutions of the planet ie $D - P = G$ where G stands for the Sāvana days of the planet. Hence $P = D - G \therefore \frac{A \times P}{M}$ where A is the Ahargaṇa, and M the number of mean solar days in a Kalpa $= \frac{AD}{M} - \frac{AG}{M}$. But $\frac{A \times P}{M} = I + F$ where I is the integral number of the revolutions of the planet and F the fractional part of a revolution which is the planetary position required. Thus from the equation $\frac{A \times P}{M} = \frac{A \times D}{M} - \frac{A \times G}{M} = I + F$. Omitting the integral part in $\frac{A \times D}{M}$ and signifying the remainder as f^1,

we have $f' - \frac{A \times G}{M} = I' + F$ where I' is some integer other than I. But $\frac{A \times D}{M}$ which gives us the number of diurnal rotations of the stars upto the day concerned is equal to the Ahargaṇa plus the number of revolutions of the Sun upto the day concerned, because Ahargaṇa + Revolutions of the Sun = diurnal rotations of the stars upto the day concerned. Omitting the integral Ahargaṇa and the integral number of revolutions of the Sun we have that the fractional part of $\frac{A \times D}{M}$ ie f' is no other than the Sun's position. Hence.

$$\frac{A \times P}{M} = I + F = f' - \frac{A \times G}{M} \text{ and}$$

$\frac{A \times G}{1577916450000}$ revolutions $= \frac{12 R}{1577916450000}$ Rasis $= \frac{R}{131493037500}$ Rasis. Thus the planetary position is equal to the Sun's position minus $\frac{A \times G}{131493037500}$ where the integral parts on either side could be ignored.

Here there is a peculiarity in this method. The planet could right away be obtained from the formula $\frac{A \times P}{M}$ where A is the Ahargaṇa, P the number of sidereal revolutions of the planet and M the number of mean Solar days in a Kalpa. Though the given method is more cumbrous than finding through the above formula, Bhaskara deliberately gives it to show the equivalence of various procedures at the same time giving us a beautiful technique as mentioned in his commentary. Thus in the above equation $\frac{A \times P}{M} = \frac{A \times D}{M} - \frac{A \times G}{M}$ the first term on the right hand side could be termed the Bha-bhramạ-graha and the second term $\frac{A \times G}{M}$ the graha-Sāvana-Dina-graha. We

have seen above that the first term is no other than the Mean Sun ignoring the integral number.

Verses 10, 11. Proof of other methods of computing planetary position. Even as the sums or differences of two or more of the numbers of Adhimāsas, Kshayāhas, lunations etc give the number of sidereal revolutions of the planets the sums or differences of two or more of the positions of the imaginary planets which go by the names Adhimāsa-graha, Kshayāhagraha etc computed out of the numbers of those Adhimāsas Kshayāhas etc. give the respective planetary positions.

Comm. This interesting concept is based upon the following principle. Suppose P to be the number of sidereal revolutions of a planet; then $\frac{A \times P}{M}$ gives the planetary position, where A is the Ahargaṇa, and M the number of mean solar days in a Kalpa. Now suppose $P = x \pm y \pm z$ where x, y, z are the numbers of Adhimāsās etc in a Kalpa, then

$$\frac{A \times P}{M} = \frac{A \times x}{M} \pm \frac{A \times y}{M} \pm \frac{A \times z}{M}.$$

The terms on the right-hand-side may be construed to be the Adhimāsa-graha etc, which are the positions of imaginary planets and their sums or differences give therefore the position of the planet, as could be seen from the above equation. Thus for example, we have the equation $P_1 - 13P_2 = a$ where P_1 is the number of the sidereal revolutions of the Moon and P_2 the number of the sidereal revolutions of the Sun, *a* stands for the Adhimāsas because Chāndramasas − Sauramāsas = Adhimāsas ; but Chāndramāsas = $P_1 - P_2$ and Sauramāsas = $12 P_2$ so that $P_1 - P_2 - 12P_2 = a$ ie $P_1 - 13P_2 = a$. From this equation

$$A \times P_1 = \underline{a \times A} + 13P_2 \times A$$

The first term on the right hand side is termed as the Adhimāsa-graha, and the second term is evidently 13 times the position of the Sun whereas the term on the left-hand-side is the Moon's position. Hence, ignoring the integral number of revolutions, we have Moon's position = Adhimāsa – graha + 13 times Sun's position (Ignoring the number of integral revolutions means subtracting integral revolutions or adding them if necessary).

Verses 12, 13. A few more examples on the aforesaid principle. The planetary position obtained by the sum of the sidereal revolutions of two planets, added to or Subtracted from another planetary position obtained by the difference of the sidereal revolutions of two planets and divided by two gives the positions of the two planets respectively, the quicker of the two in the first case and the slower in the second. Similarly the planetary position computed from the difference of the sidereal revolutions of two planets subtracted from the planetary position of the quicker of the two gives the position of the slower whereas the former planetary position added to the position of the slower gives the quicker.

Comm. We have $\frac{(P_1 + P_2) + (P_1 - P_2)}{2} = P_1$ (1) and $\frac{(P_1 + P_2) - (P_1 - P_2)}{2} = P_2$ (2) where P_1 is the number of sidereal revolutions of a quick-moving planet and P_2 that of a slow-moving one. Multiplying the above equations by $\frac{A}{M}$ with the former notation,

$$\frac{\frac{A}{M}(P_1 + P_2) + \frac{A}{M}(P_1 - P_2)}{2} = \frac{P_1 \times A}{M} \quad (3) \text{ and}$$

$$\frac{\frac{A}{M}(P_1 + P_2) - \frac{A}{M}(P_1 - P_2)}{2} = \frac{A}{M} \times P_2 \quad (4)$$

Equations (3) and (4) mean what has been stated in verse (12). Again we have the equations $P_1 - (P_1 - P_2) = P_2$

and $P_2 + (P_1 - P_2) = P_1$ where P_1 and P_2 are the numbers of sidereal revolutions of a quick and slow moving planets respectively. Following the same principle as above we could obtain their positions by multiplying the equations through out by $\frac{A}{M}$ and calling planets on the left-hand-side as (1) Dwiparyayāntarodbhava-graha subtracted from the quick-moving one and (2) the slow-moving planet increased by the Dwiparyayāntara-graha respectively.

Verse 14. The difference of the S'ighra and S'ighra-Kendra as well as the Sum of the Mandoccha and the Manda Kendra give the planet to be computed. Or again a computed planet multiplied by the number of sidereal revolutions of a planet to be computed and divided by the number of sidereal revolutions of the computed gives the planet to be computed.

Comm. $U_1 - P = K_1$ and $P - U_2 = K_2$ where U_1, P, U_2, K_1 and K_2 are respectively the number of Sidereal revolutions of the S'ighroccha, planet, the Mandoccha, the S'ighra-Kendra and the Manda Kendra; hence, we have

$P = U_1 - K_1 = U_2 + K_2$ from these equations also by multiplying througout by $\frac{A}{M}$, we have the planetary position as the difference of the S'īghroccha-graha and S'ighra Kendra-graha or the sum of the Mandoccha-graha and Manda-Kendra-graha.

Again, if P_1, P_2 be the numbers of sidereal revolutions of a computed planet and one to be computed respectively and if p_1, p_2 be the computed planet and the one to be computed, then

$$P_1 \times \frac{A}{M} = p_1,\ P_2 \times \frac{A}{M} = p_2 \text{ so that } \frac{P_1}{P_2} = \frac{p_1}{p_2}$$

$\therefore\ \frac{P_1 \times p_2}{p_1} = P_2$ which means that the position of the

planet to be computed is got by multiplying the planetary position of a planet computed by the number of sidereal revolutions of the planet to be computed and dividing by the number of sidereal revolutions of the planet computed.

Verse 15. We get the Ahargaṇa by multiplying the planetary position given in number of revolutions and fraction of a revolution by the number of days in a Kalpa and dividing by the number of sidereal revolutions in a Kalpa. How by indeterminate analysis we get the same Ahargaṇa, given the number of past sidereal revolutions alone, or by the fractional part of a revolution alone, or by the sum of the fractional parts in the case of more items involved, I shall tell later.

Comm. While computing the planet we have the formula $\frac{A \times P}{M} = p$ where A = Ahargaṇa, P = the number of sidereal revolutions of the planet, M = number of mean solar days in a Kalpa and p = the planetary position consisting of the number of past revolutions and also the fraction of a revolution. From the above equation, the Ahargaṇa $A = \frac{M \times p}{P}$ as stated. In tne case of only the integral number of revolutions or the fraction of a revolution alone being given, or the sum of remainders if more items than one are involved, the method of fiinding the Ahargaṇa is illustrated in goḷādhyāya under praśna-adhyāya under verses 12—21.

Verses 16, 17. Method of getting the time in solar years that has elapsed from the beginning of the Kalpa, given the Ahargaṇa.

The given Ahargaṇa multiplied by the number of Kshaya-tithis in a Kalpa and divided by the number of civil days in a Kalpa gives the number of the elapsed Kshaya tithis. Adding these to the Ahargaṇa we have the lunar

days L. These again multiplied by the number of Adhikamasas in a Kalpa and divided by the tithis in a Kalpa gives the elapsed number of Adhikamāsas. Multiplying this number by thirty and subtracting from the above lunar days L, we have the elapsed solar days. Dividing these by thirty, we have the number of elapsed solar months, the remainder being solar days. Dividing the solar months by 12, we have the elapsed solar years and the remainder here are the solar months. Thus we have the solar years, solar months and solar days corresponding to the given Ahargaṇa.

Comm. The inverse process detailed here is quite clear.

Verse 18. Computation of the Ahargaṇa and the planetary positions from the beginning of the Kaliyuga.

Find the Ahargaṇa from the beginning of the Kaliyuga either (according to the method described formerly with respect to a Kalpa) and this Ahargaṇa begins from Friday, Computing the mean planetary positions from this Āhargaṇa and adding to their mean positions at the beginning of the Kali which are known as Dhruvakas, we have their planetary positions for the day concerned.

• *Verses* 19, 20. The Dhruvakas of the planetary positions at the beginning of Kali, given in a tabular form.

Mars	Mercury	Jupiter	Venus	Saturn	Solar Apogee	Lunar Apogee	Ascending lunar node	
11 R	11 R	11 R	11 R	11 R	2 R	4 R	5 R	Rasis
29°	27°	29°	28°	28°	17°	5°	3°	Degrees
3′	24′	27′	42′	46′	45′	29′	12′	minutes
50″	29″	36″	14″	34″	36″	46″	58″	Seconds

Comm. The mean Sun and the mean Moon are taken to be in conjunction at the zero-point of the Zodiac. The planetary positions given above are accepted by Bhaskara on the authority of Brahmagupta. The fact that these positions differ from those given by Aryabhata signifies that Brahmagupta observed the True positions in his own time and to obtain those positions by calculation, he must have changed the fundamental constants such as the number of civil days, and sidereal revolutions of planets etc in a Kalpa. Here ends the section known as grahā-nayana.

MADHYĀDHIKĀRA - KAKSHĀDHYĀYA

Verses 1, 2. The circumference of Akāsa—Kakshā.

Astronomers say that the circumference of Akās'a-Kaksha is 18712069200000000 yojanas. Some say it is the circumference of the universe whereas some say that it is the circumference of the mountain which goes by the name Lokāloka. Those who perceive the celestial sphere as a fruit of the emblic myrobalan, (Known as Āmalaka in Sanskrit) placed in the palm, say that it is the circumference of the sphere of solar radiation ie the imaginary sphere whose volume is filled by solar light.

Comm. Bhāskara, in the course of the Commentary makes it clear that he does not subscribe to this idea which is only mythological. Look at his words which are significant and testify to his rational outlook "नाऽस्माकं मतमित्यर्थः, प्रमाण शून्यत्वात्" ie "This is not our view; because it is baseless". A yojana will be seen to be equal to 5 miles approximately.

Verse 3. He gives his personal view as follows.

The universe may be bounded or unbounded; our view is that this dimension of the circumference is no other than the distance covered by each planet in the Kalpa.

Comm. This was an assumption made by the ancient Hindu Astronomers, as well as another assumption that the distance covered by every planet during a day is the same. This we shall see later.

Verse 4. The circumference of the universe given above divided by the number of the sidereal revolutions in a Kalpa of any planet gives the circumference of the planetary orbit, so that in a Kalpa, the total distance covered is the circumference of the universe.

Comm. Clear.

Verse 5. The circumferences of the orbits of the Sun, Moon and the Stars.

The circumference of the Sun's orbit is $4331497\frac{1}{2}$ yojanas, that of the Moon 324000 yojanas, of the stellar sphere 259889850 yojanas.

Comm. Later, we are told by Bhaskara that the circumference of the earth is 4967 yojanas and its diameter is 1581. As the method given by him in the Commentary in that context, to measure the circumference of the earth is correct, we may take it that the ancient Hindu Astronomers could estimate the same correctly. If that be so, when Bhaskara gives the circumference to be 4967 yojanas, it means 4967 yoj $= \frac{3960 \times 44}{7}$ miles or 1 yojana $= \frac{3960 \times 44}{7 \times 4967} = 5.01$ miles or what is the same 1581 yojanas $= 7920$ miles ie one yojana $= 5.01$ miles approximately. With this measure of a yojana, the Moon's mean distance from the earth's centre should be (as given above)

$$\frac{324000 \times 7}{44} \times 5 \text{ miles} = 257725 \text{ miles approximately.}$$

This seems to be a fair estimate and we have to find out how this estimate could be made—Indeed, there are many elementary trigonometrical methods of finding the distance of the Moon. Which of them was used by the Hindu Astronomers, we have to discuss. Bhaskara, however takes it implicitly from Brahmagupta's version. The latter does not mention from what source he derived it but simply mentions that he has resuscitated the Brahma-Siddhānta, which grew obsolete. Either he or the author of Brahma-Siddhānta must have computed this distance using trigonometry. The following seems to be the simplest method, by which the Moon's distance was originally estimated–Refer fig. 1. Let z be the zenith-distance of the Moon as obser-

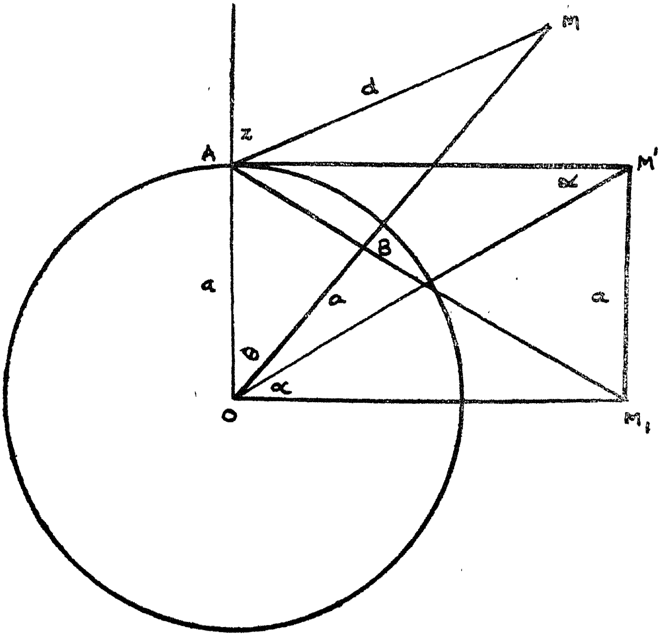


Fig. 1

ved from a place A on the primary meridian going through Lanka, Ujjain, Kerukshetra etc by an instrument (A protractor) described by Bhaskara under verse 5 Chandragrahaṇādhikāra, grahagaṇita at the time of transitting. Let B be a sublunar point on the earth at the moment of that observation, where B is also on the same primary meridian. Since both the places happen to be on the primary meridian, and such places were primarily known, both the observations could be made simultaneously. Knowing the distance A B between the two places, the angle θ subtended by A B could be easily got from the triangle A O M where O is the earth's centre and M the Moon. As a first approximation,

$$\frac{d}{\theta} = \frac{d + a}{\text{Sin } Z} = \frac{a}{\text{Sin } Z - \text{Sin } \theta}$$ taking O M roughly to be equal to (a + d). In the above equation, a being known

$d = \frac{a \text{ Sin } \theta}{\text{Sin } Z - (\text{Sin } \theta)}$. Strictly speaking $\frac{d}{\text{Sin } \theta} = \frac{a}{\text{Sin } (Z\text{-}\theta)}$

so that $d = \frac{a \text{ Sin } \theta}{\text{Sin } (Z\text{-}\theta)}$. Since $(\text{Sin } Z - \text{Sin } \theta) < \text{Sin } (Z\text{-}\theta)$ $\therefore$ the estimate of d obtained is a little greater than its true value. We shall discuss other possible methods of finding the Moon's distance in the chapter on lunar eclipses.

Having got the distance of the Moon as afore-said, it was easy to obtain the angle $\propto$ marked in fig. 1 for,

$$\text{Sin } \propto = \frac{M_1 M^1}{O M^1} = \frac{a}{d} = \frac{2\pi a}{2\pi d} = \frac{4967}{324000}$$

Since $\propto$ is small Sin $\propto$ = $\propto$ radians.

$$\therefore \propto = \frac{4967 \times 3438}{324000} = \frac{4967 \times 191}{18000} = 52.7'$$

Since the Moon's daily average motion $790' - 35''$ is had in sixty Nadis, 52.7′ of motion is covered in 52.7×60 approximately $= \frac{60}{15} = 4$ Nadis. Thus the fact that we are given the horizontal parallax as 4 Nadis in the context of lunar eclipses is based on this.

Having obtained thus the distance of the Moon from the centre of the earth ε, and having measured the angular diameter of the Moon's disc with the help of the protractor mentioned above, from the triangle εAM (Ref. fig. 2) where

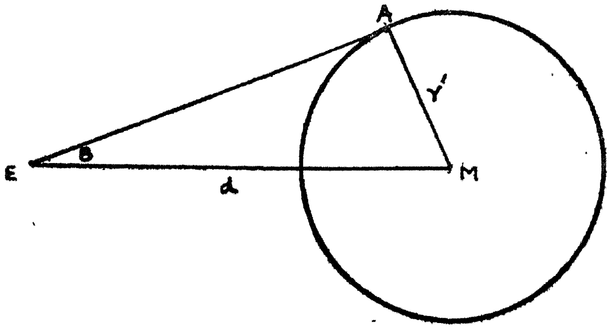


Fig. 2

εA is a tangent to the Moon's disc and M the centre of the Moon, $\text{Sin B} = \frac{r'}{d}$ where B is the angular semi-diameter of the Moon's disc, r' = the spherical radius of the Moon's disc measured in yojanas, so that since Sin B=B' in Hindu trigonometry when the angle is small,

$$\frac{324000}{} \quad \frac{3438}{} \quad \therefore 2\pi r' = \frac{324000 \times 16}{3438}; \text{ hence}$$

$$r' = \frac{18000 \times 16}{191} \times \frac{7}{44} = \frac{504000}{2101} = 240 \text{ yojanas.}$$

Bhaskara gives the semi-diameter of the Moon's disc to be $16'-0''-9'''$ so that the above r' is very approximately 240 yojanas as given by Bhaskara.

Now with these constants pertaining to the Moon's distance, and his spherical radius, it was sought to find the spatial distance traversed by the Moon during a day. The rule of three used was "If $16'-0''-19'''$ of the Moon's angular semi-diameter pertains to a spatial distance of 240 yojanas at his orbit, what does the mean daily motion of $790'-35''$ pertain to?" The answer is $\frac{683064000}{57609} = 11858\frac{3}{4}$ yojanas as given by Bhaskara elsewhere or taking the Moon's mean semi-angular diameter to be $16'$ only, and using the rule of three "If $16'$ at the lunar orbit correspond to 240 yojanas, what do $790'-35''$ correspond to" we have $\frac{790'-35'' \times 240}{16'} = 15 \times 790\frac{7}{12} = \frac{5}{4} \times 9487 = 9487 +$
$= 11858\frac{3}{4}$ yojanas as given by Bhaskara.

Thus having obtained the daily spatial motion, the Hindu Astronomers assumed all the other planets (including the Sun) to have the same daily mean spatial motion. On this assumption since the, Moon's orbit will be $27.3217 \times 11858\frac{3}{4} = 324000$ yojanas, and the circumference of the universe will be $324000 \times 57753300000 = 18712069200000000$ yojanas the Sun's orbit will be

$$\frac{\text{circumference of the universe}}{\text{Number of sidereal revolutions}} = 4331497\tfrac{1}{2} \text{ yojanas}$$

$$\left(\text{because } \frac{324000 \times 57753300000}{4320000000} = \frac{3240}{432} \times 577533\right.$$

$$\left. = \frac{15}{2} \times 577533 = \frac{8662995}{2} = 4331497\tfrac{1}{2}\right).$$

Also, the circumference of the stellar universe was presumed to be sixty times that of the Sun's orbit.

With the constants obtained for the Moon's diameter and distance and with assumption that all the planets have the same daily motion, the constants pertaining to the Sun and the other planets were obtained. We shall resume this topic in another context.

Verse 6. The mean daily motion of the planets.

The circumference of the universe divided by the number of days in the Kalpa, gives the daily spatial motion of a planet. The planets move thus a distance of $11858\frac{3}{4}$ yojanas in a day.

Comm. Already explained.

Verse 7, *and half the verse* 8. The Ahargaṇa multiplied by 11859 decreased by the quotient obtained by dividing the product of the Ahargaṇa and 9921 by 35419 gives the distance covered by a planet in yojanas. These yojanas divided by the circumference of the planet's orbit gives the fraction of a revolution and the integral number of revolutions made.

Comm. Let the Ahargaṇa be A; every planet should have described a space equal to A×D where D is the distance traversed per day and is just a little less than 11859, or more correctly should have described a space equal to $A \times \frac{C}{c}$ where C is the afore-said circumference of the

universe, and c the number of civil days or mean solar days in a Kalpa. For the sake of an easy computation Bhaskara gives here an interpolation. In the first place, we are asked to multiply A by 11859 by which, an excess is there in the result; then this excess is sought to be removed. The excess is $A \times 11859 - \frac{A \times C}{c}$ because $A \times \frac{C}{c}$ is the correct distance described. In other words, the distance described by any planet is

$$\frac{A \times C}{c} = A \times 11859 - \left(A \times 11859 - \frac{A \times C}{c}\right) \qquad \text{I}$$

But $\frac{C}{c} = \frac{1871206920000000}{1577916450000} = \frac{420024000}{35419}$ dividing by the common factor 44550000

∴ The space described by any planet is from I

$$A \times 11859 - A \left\{\frac{c \times 11859 - C}{c}\right\}$$

$$= A \times 11859 - A \left\{\frac{35419 \times 11859 - 420024000}{35419}\right\}$$

But the numerator within the brackets is 9921

∴ Space described $= A \times 11859 - \frac{A}{-} \times$

as stated. This distance divided by the individual orbital lengths, we have the integral number of revolutions made by each planet, rejecting which we have the fractional part of a revolution which gives the position of the planet.

Verses 8, 9. The orbit of the planet itself is no doubt the orbit of the Mandoccha (apogee with respect to the Sun and the Moon and aphelion with respect to the other Star-planets ie Mars, Mercury, Jupiter, Venus and Saturn) and of the node (point of intersection of the orbits of the Star-planets with the ecliptic); but while Computing the positions of these Mandocchas and the Nodes, as per the method indicated above ie according to the method of

Kakshādhyāya, their orbits are taken to differ from those of the planets. (because slow-moving points will have longer orbits as per the assumption made namely that the circumference of the universe divided by the number of the sidereal revolutions gives the length of the orbit). Similarly the orbit of the Sun itself will be the orbit of Mercury and Venus, and the orbits of their S'ighrocchas are their real orbits wherein Mercury and Venus are taken to move with the velocity of the Sun.

Comm. The prescription of the computation of a planetary position as per the method of Kakshādhyāya, has brought in an awkward situation. Let us consider the Computation of the positions of Mercury and Venus. These two planets oscillate about the mean position of the Sun, because their orbits happen to lie within the earth's orbit. Hence their mean sidereal periods coincide with that of the Sun, which means that their numbers of sidereal revolutions coincide with the number of sidereal revolutions of the Sun. Hence the Kakshādhyāya method of Computing a planetary position brings in the idea that the orbits of Mercury and Venus coincide with the orbit of the Sun. Bhāskara perceived the awkwardness of this situation and says therefore, the above coincidence of the orbits must not be taken to be a reality but is intended only for the sake of computation. The actual planets Mercury and Venus in fact revolve says Bhaskara in the orbits of their Sīghrocchas, with the velocity of the Sun-Even this supposition that the actual planets move with the velocity of the Sun sounds odd, but this will be clarified later in the spashta-adhikāra, wherein we propose to explain the peculiar concept of S'īghroccha at length.

Here ends the Grahānayanādhyāya according to the Kakshādhyāya method.

THE ADHYĀYA KNOWN AS PRATYABDA S'UDDHI IN MADHYĀDHIKĀRA

Verse 1. The number of years which have elapsed from the beginning of the Kalpa, respectively multiplied by 2, 4 and 3 and divided by 8, gives what is called Dinādya in days, ghatis and Vighatis respectively. If this be added to the number of years and divided by seven, the remainder gives the Abdapa or the lord of the year, (under whose name the week-day of the commencement of the year stands).

Comm. One mean solar year consists of 365 days, 15 ghatis, 30 palas, and $22\frac{1}{2}$ Vipalas where the units are all mean solar and one mean solar day is equal to 60 mean solar ghatis, one mean solar ghati is equal to 60 mean solar palas, one mean solar pala is equal to 60 mean solar Vipalas and so on in sexagesimal sub-division. The fraction of the day over and above 365 days, namely 0–15–30–22–30 multiplied by 8 gives 2 days, 4 ghatis and 3 palas so that by the rule of three i.e. 'If in 8 mean solar years, the fraction accrues to 2 days, 4 ghatis and 3 palas, what will it accrue to in x elapsed mean solar years from the beginning of the Kalpa? we have the answer $\frac{x \times 2}{8}$ days, $\frac{x \times 4}{8}$ ghatis and $\frac{x \times}{8}$ palas. If this is added to the number of the elapsed years, and the result divided by seven, the remainder gives the week-day of the commencement of the concerned year, because the remainder got by dividing 365 by 7 is one, and the week day advances at the rate of one per year. Also the Kalpa began on Sun-day.

Verse 2. Alternate method.

Half the number of elapsed years added to $\frac{1}{80}$ of itself, then divided by 60 and added to $\frac{1}{4}$ of the elapsed years, gives the Dinādya.

Comm. Since the Dinādya per year is 0–15–30–22–30, in x elapsed years it accrues to x × 0–15–30–22–30 $= \frac{x \times 15}{60}$ days $+ \frac{x \times 30}{60}$ ghatis $+ x \times 22\frac{1}{2}$ palas $= \frac{x}{4}$ days$+$ $\frac{x}{2}g + \frac{x \times 45}{2 \times 60}$ palas $= \frac{x}{4}d + \frac{x}{2}g + \frac{3x}{8} \times \frac{1}{60}g = \frac{x}{4}d +$ $\frac{x}{2}(1+\frac{1}{80})\,g = \frac{x}{4}d + \frac{x}{2}\frac{(1+\frac{1}{80})}{60}d$ which is the given formula.

Verse 2. Alternative method.

The number of elapsed years divided by respectively 4, 120, and 9600 and the Sum taken gives the Dinādya.

Comm. Let x be the number of elapsed years. The Dinādya as before is x × 0–15–30–22–30 $= \frac{x}{4}d + \frac{x \times 30}{60 \times 60}d$ $+ \frac{x \times 45}{2 \times 60} \times \frac{d}{60 \times 60} = \left(\frac{x}{4} + \frac{x}{120} + \frac{x}{9600}\right)d$ as given.

Verse 3. To obtain what is known as Kshayāhādya. The number of elapsed Kshayāhās from the beginning of Kalpa upto the commencement of the year, is obtained as follows. Let x be the number of elapsed years; then $x - \frac{x(1+\frac{1}{60}) + 30x}{160}$ = Kshayāhās.

Comm. The number of Kshayāhās in a Kalpa of 4320000000 solar years is 25082550000 so that per year their number is 5–48–22–7–30. In this 0–48–22–7–30 is said to be Kshayāhādya per year = 1 − (0–11–37–52–30) putting the quantity within the brackets into a fraction, $52\frac{1}{2}$ vipalas $= \frac{105}{2} \times \frac{1}{60} = \frac{7}{8}$ palas; $(37+\frac{7}{8})$ palas $= \frac{101}{8} \times \frac{1}{60}$ ghatis $= \frac{101}{160}$ ghatis ; $11\frac{101}{160}$ ghatis $= \frac{1861}{160 \times 60}$ days

$\frac{}{160}$ of $\frac{1861}{60}$ days $= \frac{1}{160}$ of 31 days, one ghati. Hence per year the Kshayāhādya is $1 - \frac{1}{160}$ of $\overset{d}{(31}\text{-}\overset{g}{1)}$ so that for x years it would be $\left\{x - \frac{x}{160}\left(\overset{d}{31}\text{-}\overset{g}{1}\right)\right\}$ d

$$= x - \frac{1}{160}\left\{30x + x\left(1 + \tfrac{1}{60}\right)\right\}$$ which is the formula given.

4. Alternative method.

The Dinādya obtained before multiplied by three, is to be diminished by $\frac{1}{400}$th of the number of years; the result increased by $\frac{1}{30}$th of the number of years gives the number of Kshayāhās.

Comm. The Dinādya pertaining to one year is 0–15–30–22–30 and the Kshayāhādya is 0–48–22–7–30. Multiply the former by 3 and subtract from the latter; we have $0\text{-}1\text{-}51 = 0\text{-}1\text{-}\frac{51}{60} = 0\text{-}1\text{-}\frac{17}{20} = 0\text{-}\frac{37}{20} = \frac{37}{20} \times \frac{1}{60} = \frac{37}{1200}$ day Hence $K - 3D = \frac{37}{1200}$ day where K = Kshayāhādya and D = Dinādya $\therefore K = 3D + \frac{37}{1200}$ hence per x years it

$$= 3D \times x + \frac{x}{30} - \frac{x}{400}$$ which is the formula given.

4 contd. Alternative method.

Comm. The Kshayāhādya for an year is 0–48–22–7–30 48 ghatis $= (1 - \frac{1}{5})$ day; hence for x years $x(1 - \frac{1}{5})$.

The remaining fraction $= 0\text{-}0\text{-}22\text{-}7\frac{1}{2} = 0\text{-}0\text{-}(22+\frac{15}{2}\times\frac{1}{60}$

$$= 0\text{-}0\text{-}22\tfrac{1}{8} = 0 - \frac{177}{8} \times \frac{1}{60} = 0 - \frac{59}{160} = \frac{60-1}{160 \times 6}$$

$$\text{day} = \frac{1}{160} - \frac{1}{60 \times 160} = \frac{1}{160}\left(1 - \frac{1}{60}\right);$$ hence for :

years $\frac{x}{160}\left(1 - \frac{1}{60}\right)$

Adding the two we have the required formula.

Verse 5. To obtain the elapsed number of Adhikamāsās and what is called S'uddhi.

The sum of the Dinādya, Kshayāhādya and ten times the number oi elapsed years divided by 30, gives the number of the elapsed Adhikamāsas; the remainder is known as S'uddhi if diminished by the fraction of the Kshayāhas.

Comm. The number of mean solar days in an year is 365-15-30-22-30; the Kshayāhās in an year are 5-48-22-7-30; adding the two we have the number of tithis in an year equal to 371-3-52-30. The number of solar days being 360, the number of tithis which constitute the Adhimāsas is equal to 11-3-52-30. The sum of the Dinādya and Kshayāhādya in an year = 0-15-30-22-30 + 0-48-22-7-30 = 1-3-52-30. Hence the above number of tithis which constitute the Adhimāsas namely 11-3-52-30 = 10 + Sum of Dinādya and Kshayāhādya pertaining to an year. Hence for x years, the Adhimāsa days are equal to x×10+Sum of Dinādya and Kshāhādya for x years. These Adhimāsa days divided by 30 give the Adhimāsas, and the remainder is called S'uddhi, so called because while finding the Ahargaṇa from the beginning of a solar year, this remainder has to be subtracted. Why the fraction of Kshayahas, which is there in this S'uddhi is prescribed to be subtracted, will be explained in another context.

Verse 6. An alternative method to find the Adhimāsas. The number of years divided separately by 32 and 30,

the sum increased by the number of years multiplied by eleven and the result divided by 30 gives the number of elapsed Adhimāsas. The remainder diminished by the fraction of Kshayāhās as mentioned before, is the S'uddhi.

Comm. Every year, the number of Tithis that constitute the Adhimāsās accruing in the course of years, is 11-3-52-30. The fractional part of this namely 0-3-52-30

$$= 0\text{-}3\text{-}52\tfrac{1}{2} = 0 - 3 + \frac{105}{2\times 60} = 0 - 3\tfrac{7}{8} = 0 - \frac{31}{8}$$

$$= \frac{31}{8} \times \frac{1}{60} = \frac{31}{480} = \frac{16+15}{480} = \frac{1}{30} + \frac{1}{32}$$ converted

into days. Hence for x years $\frac{x}{30} + \frac{x}{32}$. Adding the

number of days 11x, we have $11x + \frac{x}{30} + \frac{x}{32}$ as stated.

Dividing these days by 30, we have the number of elapsed Adhimāsas and the remainder is S'uddhi, if the fraction of the Kshayāhās is subtracted therefrom as indicated.

Verse 7. Method of finding the lord of the year without a knowledge of Dinādya.

The week-day at the Commencement of the Solar year, which pertains to the lord of the year, is the remainder got by dividing by seven the S'uddhi which is itself diminished by the remainders got by dividing by seven separately firstly double the excess of the elapsed years over the elapsed Adhikamāsās and secondly the elapsed Kshayāhās. This may be put in symbols as $R_7 \{ S—R_7 (2y—A) — R_7. K. \} = R_7 \{ S' — R_7 (2y — 2A + K) \}$ where R_7 signifies the remainder got by dividing by seven the quantity which follows, S signifies S'uddhi, y means the elapsed Solar years A the elapsed Adhikamāsās and K the elapsed Kshayāhās.

Comm. The lord of the year is the lord of the week-day at the Commencement of the solar year. To get this week-day we have if the number of Tithis elapsed upto the beginning of the luni-solar year be T, the Suddhi S′, the Kshayāhās upto the Commencement of the solar year be K, then since the number of civil days is equal to T + K, R_7 (civil days) = R_7 (T + S — K). But T = 360y + 30A where y is the number of elapsed solar years and A the number of elapsed Adhikamāsās. ∴ R_7 (T) = R_7 (360y + 30A) = R_7 (3y + 2A) since the remainder got by dividing 360 and 30 by seven are respectively 3 and 2. Hence R_7 (civil days = Ahargaṇa) = R_7 (3y + 2A) + R_7S′ — R_7K. The number of Kshayāhās K = y (5-48-22-7-30) = 5y + y (0-48-22-7-30). But y (0-48-22-7-30) = K where K is the previously calculated Kshayāhādya. Hence R_7K = $R_7$5y + R_7K. Substituting this in the above R_7 (civil days) = R_7 (3y + 2A) + R_7S′ — R_7 (5y + K) = R_7S′ — R_7 (5y — 3y) + R_7 (2A) — R_7K = R_7 (S′ — R_7 (2y — 2A) — R_7K] which is the given formula = R_7 (S′ — R_7 (2y — 2A + K)].

Verse 8. To obtain the fractional part of the elapsed Kshayāhās even without a knowledge of the number of Kshayāhās.

The excess of the ghatis of the Adhimāsa Sesha ie the remainder got by dividing the sum of the Dinādya, Kshayāhādya and ten times the number of elapsed years by 30 as indicated in verse 5, over the ghatis of the Dinādya, obtained under verse 1 gives the fractional part of the Kshayāhās.

Comm. The ghatis or the fractional part of the masa Sesha is the sum of the fractional parts of Dinādha and the Kshayāhādya so that the excess of the ghatis of the Adhimāsa Sesha over the ghatis or the fractional part of the Dinādya gives the ghatis or the fractional part of the Kshayāhās.

Verse 9. The planetary positions at the end of the elapsed solar year.

The number of the elapsed solar years multiplied by the number of sidereal revolutions of the respective planets in a Kalpa and divided by the number of Solar years in a Kalpa gives the planetary positions at the end of the last Solar year.

Comm. This is a simple rule of three. The planetary positions so got, leaving out the integral numbers of revolutions made which are not required, are called the Dhruvakas of the respective planets for the ensuing solar year. With respect to the apogee of the Sun and the aphelia and nodes of the star—planets these Dhruvakas themselves give their positions for the whole ensuing year, as their motion is very very slow.

Verse 10. An alternative method of obtaining the Dhruvaka of the Moon.

The Adhimāsa Sesha multiplied by 12 gives the position of the Moon at the Commencement of the Solar year.

Comm. Since the position of the Sun is at the Zero-point of the Zodiac at the Commencement of the solar year, and since the Adhimāsa Sesha is the difference of the solar and luni-Solar systems of reckoning, or what is the same, the arc gained by the Moon over the Sun, which is no other than the elongation of the Moon measured in Tithis each tithi being of 12° gain of elongation, so the Adhimāsa Sesha at the Commencement of the solar year in Tithis multiplied by 12 gives the longitude of the Moon at that Commencement. (Note - Bhaskara waxes into poetic eloquence in the second half of the verse, having given the procedure in the first half).

Verse 11. Procedure prescribed in the event of computing planetary positions from the beginning of the Kaliyuga for the sake of convenience.

The Dinādya may be also obtained from the beginning of the Kali, which begins with Friday. The Dhruvas calculated for the commencement of the solar year are to be added to the planetary positions at the beginning of the Kali, in the event of computing the Ahargaṇa and thereby the planetary positions from the beginning of the Kaliyuga for the sake of convenience.

Comm. The Dinādya at the commencement of Kali is Zero, since the number of civil days during the length of time equal to a Kaliyuga is integral and equal to 157791645 according to Brahmagupta and Bhaskara who follows him. The Suryasiddhānta, it may be noted here, gives the number of days in a Kaliyuga as not integral but equal to 157791782.8. Hence the Dinādya computed from the beginning of the Kali is to be increased by .8 to obtain its value according to Suryasiddhānta. The planetary positions at the commencement of Kali were given by Bhaskara already.

Verse 12. The number of what are called Kshepadināś to obtain the Ahargaṇa.

Hereafter Bhaskara is going to obtain the planetary positions for any day during the current solar year having obtained the Dhruvakas or the planetary positions for the beginning of the Solar year. In that behalf the Ahargaṇa or the collection of days which have elapsed from the commencement of the Solar year is to be found. This Ahargaṇa is obtained by subtracting the number of Kshayāhās from the number of tithis that have elapsed. In finding these Kshayāhās, we have to take note that there is a little remnant of Kshayāhās at the beginning of the Solar year which is also to be taken into account while computing the number of Kshayāhās during the course of the year. In other words, the number of Kshayāhās that are going to be computed during the course of the year, for the elapsed part of the year will be in default of the

actual number if we ignore the accrued fraction of Kshayāhās at the commencement of the Solar year. To make amends for that default we have to add some number to the numerator of the improper fraction which is going to give us the number of the elapsed Kshayāhās during the course of the year. The formula that is going to be used to obtain the number of Kshayāhās during the elapsed tithis is $\frac{x}{64}\left(1+\frac{1}{702}\right)$. This formula arises out of the fact that there are 55739 Kshayāhas during 3562220 tithis, so that for 64 tithis the number of Kshayāhās is equal to

$$\frac{64 \times 55739}{3562220} = \frac{3567296}{3562220} = 1 + \frac{5076}{3562220} = 1 + \frac{1}{\frac{3562220}{5076}} = 1 + \frac{1}{702}.$$

Let the fraction of Kshayāha at the commencement of the solar year, to be taken into account be x ghatis i.e. $\frac{x}{60}$ of a tithis (for Kshayāhās are are computed out of tithis) i.e. $\frac{x \text{ tithis}}{60}$. Let y be the number of tithis elapsed after the commencement of the solar year. Then to compute the Kshayāhās that ensue after the commencement of the solar year upto the day concerned during the course of the year, the formula to be used is $\frac{y}{64}\left(1+\frac{1}{702}\right)$. To this we have to add $\frac{x \text{ tithis}}{60}$ as the balance of Kshāyāha at the commencement of the solar year to be taken into acccount

$$\frac{x \text{ tithis}}{60} = \frac{x \times 64}{64 \times 60} = \frac{x \times \frac{64}{60}}{60}$$

Now, in computing the number of tithis which have elapsed after the commencement of the Solar year, we subtract S'uddhi from the number of tithis that have elapsed after the beginning of the luni-Solar year. But

in this S'uddhi we have subtracted the fractional part of the Kshayāhas for a different purpose so that in subtracting the S'uddhi, we have increased the tithis by the fractional part of the Kshayāhās. This increase therefore should be nullified, which means that the quantity to be added is

$$\frac{x \times \frac{64}{60}}{64} - \frac{x}{60} = \frac{x \times \frac{63}{60}}{64} = \frac{21}{20}\frac{x}{64} = \frac{x\left(1 + \frac{1}{20}\right)}{64}$$

Hence the Kshepadinas or the tithis to be added to the number of tithis which have elapsed from the commencement of the Solar year, are $x\left(1 + \frac{1}{20}\right)$. This means that the ghatis x which form the fractional part of the Kshayāhas are to be increased by one twentieth part of themselves and are to be viewed as tithis and not ghatis as mentioned.

Verse 12 (contd.) and *Verse* 13. To find the Ahargaṇa, ie the collection of days which have elapsed from the commencement of the solar year.

The number of tithis which have elapsed from the commencement of the luni-Solar year diminished by the S'uddhi, increased by $\frac{1}{702}$th part of the result, and then increased by the Kshepa-tithis aforesaid, and the result divided by 64, gives the number of Kshayāhās, which have elapsed from the beginning of the Solar year. These are to be subtracted from the tithis which have elapsed from the beginning of the Solar year to give the Ahargaṇa.

Fig. 3

Comm. (Refer Fig. 3). Let T_1 be point indicating the commencement of luni-Solar year; let R_1 be the point indicating the next Sun-rise. Let S be the beginning point of the Solar year, R_2 the preceding Sun-rise and T_2 the ending moment of the preceding tithi; Let R_3 be the

next Sun-rise, T_3 the ending moment of the tithi preceding the Sun-rise R_4 upto which point the Ahargaṇa has to be found from the commencement of the Solar year. Thus the Ahargaṇa to be found is $SR_4 = ST_3 + T_3 R_4$. Here $T_3 R_4$ is the Kshayāghatis, at the Sun-rise concerned. SR_3 is a fraction of a day that is to be there in the Ahargaṇa SR_4 we are seeking. While subtracting the S'uddhi $T_1 S$ diminished by the Kshayāghatis $T_2 R_3$ from the number of elapsed integral number of tithis from the commencement of the luni-Solar year namely $T_1 T_3$ we have $T_1 T_3 - (T_1 S - T_2 R_3) = T_1 T_3 - (T_1 T_2 - SR_3) = T_2 T_3 + SR_3$. Thus instead of $SR_3 + R_3 R_4$ we have by the above procedure $SR_3 + T_2 T_3$. Though both $T_2 T_3$ and $R_3 R_4$ are integral numbers and should be the same if the interval is small, they may differ by an integral number, if the interval happens to be long. Bhaskara says that this difference of an integral number will be rectified by the Kshepadinas found under verse No. 12, for, from these Kshepadinas, the Kshayāhas are found and subtracted from the tithis. It will be noted that the difference of an integral number between $T_2 T_3$ and $R_3 R_4$ is no other than Kshayāhas.

Or, this may be seen in an other way. We are to find $SR_4 = T_1 R_4 - T_1 S = T_1 T_3 + T_3 R_4 - T_1 S = T_1 T_3 - (T_1 S - T_3 R_4)$ = No. of elapsed tithis — (S'uddhi — Kshayaghatis at the day concerned). But instead of the Kshayaghatis at the day concerned Bhaskara prescribes the subtraction of the Kshayāghatis at the end of the Solar year ie instead of subtracting $T_3 R_4$ it is prescribed to subtract $T_2 R_3$. This difference, Bhaskara says is made up by taking into account the Kshepadinas pertaining to the Kshayāhas.

Verse 14. In case the Ahargaṇa is required for a day preceeding the commencement of the Solar year, then the elapsed tithis are less than the Tithis of the Suddhi; so subtraction is not feasible. In this case, it is prescribed

to take the elapsed tithis from the previous Chaitra and the S'uddhi of the previous year. Also in this case the Dhruvakas pertain to the commencement of the previous Solar year.

Comm. Easy.

Verse 15. To obtain the position of the Sun.

The number of the days in the Ahargaṇa is to be diminished by $\frac{1}{60}$ part of itself to obtain the number of degrees, and fractions thereof; then the Ahargaṇa multiplied by three and divided by 22 gives the minutes and fractions thereof. Adding the two results we get the position of the Sun.

Comm. The mean daily motion of the Sun is 0-59-8-10-21. Here $59' = 1^\circ - 1' = (1 - \frac{1}{60})^\circ$; for x days $x\,(1 - \frac{1}{60})^\circ = x^\circ - \frac{x^\circ}{60}$ as mentioned. The remaining part namely $0\text{-}0\text{-}8\text{-}10\text{-}21 = \frac{9807'}{72000}$; Converting this into a continued fraction it is equal to $\frac{1}{7+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{12+}\cdots$ of which a good convergent is $\frac{3'}{22}$. Hence for x days $\frac{3x}{22}$ as stated in the verse.

Verse 16. To obtain the position of the Moon.

The number of elapsed integral tithis multiplied by 12 and added to the Sun's position in degrees, gives the Moon's position in degrees at the ending moment of the tithi preceeding the day at the Sun-rise of which the planetary positions are sought. To find the position at the Sun-rise required, ten times the Kshaya-dina-Sesha increased by $\frac{1}{7}$th of itself gives the number of minutes to be added to the position got above.

Comm. The first part is clear, because for every tithi, there will be an increase of elongation of 12°. To get the fractional part of the tithi in between the ending moment of the tithi and the subsequent Sun-rise, the interval which is Sāvana ie expressed in the unit of civil day has to be converted into luni-Solar units. The rule of three is 'If for 63 Sāvana days there are 64 tithis, what will it be for the above interval?' Here the approximate ratio of $\frac{64}{63}$ is used because the Ahargaṇa which is Laghu is small ie is less than 365. The Kshaya-dina-Sesha which was got under verses 12, 13, has a divisor of 64, and is of the form $\frac{y}{64}$.

Hence the quantity to be added is $\frac{y \times 64}{64 \times 63} = \frac{y}{63}$ of a tithi

$= \frac{y}{63} \times 12 \times 60$ minutes of arc $= \frac{8}{7} \times 10\,y = 10\,y\,(1 + \frac{1}{7})$ minutes as given in the verse. This has to be added to 12t°, got before where t is the number of elapsed tithis.

Verse 17. Computation of Mars.

The mean daily motion of Mars is 0–31–26–28–7=0–30 + 0–0–90 minus 0–0–3–31–53. If x be the number of days $x \times 0\text{–}30' = \frac{x^\circ}{2}$; $x \times 0\text{–}0\text{–}90'' = \frac{x}{2} \times 3'$; thus the rule prescribed is " Half of the number of days gives the degrees; half the number of days multiplied by 3 gives the number of minutes" from this we have to subtract x × (0–0–3–31–53). The quantity within the brackets is approximately $\frac{1}{17}$ of a minute for $\frac{1}{17}' = 0\text{–}0\text{–}3''\text{–}31'''\text{–}46''''$. Since the Ahargaṇa is small the precision required is there. Thus the rule is $\frac{x^\circ}{2} + \frac{x}{2} \times 3' + \frac{x'}{17} +$ the Dhruvaka at the beginning of the Solar year.

Verse 18. Computation of the S'ighroocha of Mercury.

The mean daily motion of the Budha-S'ighra is 4°–5′–32″–18‴–28⁗.

Rule. If the Ahargaṇa be x days then the S'īghra will be $4x^\circ + \frac{4x \times 3'}{130}$ + Dhruvaka. *Proof.* For x days, the mean motion is $x \times 4^\circ + x \times (0\text{-}5'\text{-}32''\text{-}18'''\text{-}28'''')$. The second part is taken to be $4x \times \frac{3^\circ}{130}$ ie $x \times \frac{12^\circ}{130} = x \times \frac{6}{65}^\circ$ for $\frac{6}{65}$ of a degree = $0\text{-}5'\text{-}32''\text{-}18\text{-}'''28''''$. The fraction $\frac{6}{65}$ can be seen to be a convergent of the remainder for $5'\text{-}32''\text{-}18'''\text{-}28'''' = \frac{299077^\circ}{3240000} = \frac{1}{10+} \frac{1}{1+} \frac{1}{4+} \frac{1}{1+} \frac{1}{9968}$ of which the penultimate convergent is $\frac{6}{65}$.

Verse 19. The Ahargaṇa divided by 12 and by 71 gives respectively the positive degrees and negative minutes to be added to the Dhruvaka of Jupiter.

Comm. If x be the Ahargaṇa $\frac{x^\circ}{12} - \frac{\bar{x}}{71}$ + Dhruvaka = Guru.

The mean daily motion of Jupiter is 0–4–59–9.9=0–5′ minus 0–0–0–50–51. Evidently $0\text{-}5' \times x = \frac{x^\circ}{12}$. The remainder $0\text{-}0\text{-}0\text{-}50\text{-}51 = \frac{61'}{4320}$ of which the continued fraction is $\frac{1}{70+} \frac{1}{1+} \frac{1}{4}$. A very approximate convergent is $\frac{1}{71}$ as given.

Verse 19 contd. To obtain the position of the S'īghra of venus.

The Ahargaṇa multiplied by 10 and divided by 6 and 155 respectively gives degrees positive and negative to be added to the Dhruvaka of Venus to give his position.

Comm. The formula is $\frac{10\,x}{6} - \frac{10\,x}{155}$. The mean daily motion of the S'īghra of Venus is $1^\circ\text{-}36'\text{-}7''\text{-}44'''\text{-}35''''$ =

$1°-40'$ minus $0-3'052''-15'''$ approximately. For x days $x \times 1\frac{2°}{3} = \frac{5}{3}x° = \frac{10x°}{6}$ which gives the first part. Also $3'-52''-15''' = \frac{929°}{14400} = \frac{1}{15+}\frac{1}{1+}\frac{1}{1+}\frac{1}{464}$ of which a proximate convergent is $\frac{2}{31}=\frac{10}{155}$ ie for x days $\frac{10x°}{155}$. Hence the position is given by $\frac{10x°}{6} - \frac{10x°}{155} +$ Dhruvaka.

Verses 20. The position of Saturn is given by $\frac{2x'}{5} + \frac{2x''}{5} +$ Dhruvaka $=$ Saturn's position.

Comm. The mean daily motion of Saturn is 0-2-0-22-51.

For x days $2x' + x \times 0''-22'''-51''''$. The latter part is $\frac{137''}{360}$ approximately $= \frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2}$ of which a near covergent $\frac{2}{5}$ so that for x days, we have $2x' + \frac{2x''}{5}$ $+$ Dhruvaka position of Saturn.

Verses 20 contd. To find the position of the apogee of Moon.

The Ahargaṇa divided successively by 10 and 88 and added gives the degrees to be added to the Dhruvaka of the Apogee of the Moon to obtain its position.

Comm. The mean daily motion of the apogee of the Moon is 0-6-40-53-56. At the rate of $6'$ per day, in x days the number of degrees covered is $\frac{x}{10}$; the remainder

$$\frac{4601°}{\quad} = \frac{1}{88+}\frac{1}{41} \text{ of } -$$

approximate convergent is $\frac{1}{88}$. Hence the position is given

by $\frac{\triangle}{10} + \frac{x^\circ}{88}$ + Dhruvaka as given by the verse.

Verse 21. To obtain the position of the lunar Node

The Ahargaṇa multiplied by 30 and divided by 566 gives the number of degrees to be added to the Dhruvaka to give the position of Rahu.

Comm. If the number of the sidereal revolutions of Rahu in a Kalpa multiplied by twelve divide the number of days in a Kalpa we have very approximately 566, which means that Rahu traverses a Rasi very nearly in 566 days. Hence dividing the Ahargaṇa by 566, and multiplying by 30, we have the degrees covered, which being added to the Dhruvaka gives the position of Rahu.

22, 23, 24. Alternative method of obtaining the planetary positions.

The Ahargaṇa multiplied by 100000, and divided successively by 101461, 151787, 190833, 24436, 1203400, 62416, 2990000, 898000, 1886800 gives the respective positions of theplanets beginning from the Sun and those of the apogee and Node of the Moon; in the case of the Moon, however, the result is to be multiplied by 20. The results in degrees added to the Dhruvakas give their positions.

Comm. The degrees covered by the planets etc in D days are $\frac{R \times 360 \times D^\circ}{M}$ where R is the number of sidereal revolutions in a Kalpa, M the number of mean solar days in a Kalpa, and D the Ahargaṇa. Now

$$\frac{R \times 360 \times D}{M} = \frac{R \times 360 \times D \times 100000}{M \times 100000} = \frac{D \times 100000}{\frac{M \times 100000}{R \times 360}}.$$

Then $\frac{M \times 100000}{R \times 360}$ is found for every planet. In the case

of the Moon, however a multiplier 20 also is used in addition to 100000, because he has such a quicker motion.

Verses 25, 26. To obtain the mean daily motion of the planets.

The number of minutes of arc moved by a planet per day gives the mean daily motion of that planet. Though, however, the spatial velocity of each planet is the same per day, in angle, the velocities differ, (on account of the varied distances) and so we perceive slowness or fastness in the movement of the planets.

Comm. Easy.

The assumption of equal daily spatial velocities for all the planets has been explained before. The daily angular velocity of a planet in minutes of arc is $\frac{R \times 360 \times 60'}{M}$ where R is the number of its sidereal revolutions and M the number of mean solar days in a Kalpa.

Verse 27. The reason for unequal angular velocity of the planets.

Since the planetary orbits are all construed as comprising of 360° alone, a minute of arc of a smaller orbit has a smaller spatial distance which will be covered more rapidly, the velocity being constant, whereas of a longer orbit, a minute of arc means a longer distance which will be covered in a longer time at the same spatial velocity, which means that the planet appears to be slow in motion. Thus the Moon, the Mercury, Venus, Sun, Mars, Jupiter, Saturn being placed at longer distances from the earth in ascending order their orbits are longer in ascending order so that they are slower in angular motion in that order.

Here ends the section called Pratyabda S'uddhi in the chapter Madhyādhikāra.

THE SECTION KNOWN AS ADHIMĀSĀDI-NIRNAYA IN Ch. 1

Verse 1. A special feature of the computation of Ahargaṇa. If the Ahargaṇa is to be increased or decreased by unity to adjust it with the week-day, the tithis also are to be increased or decreased by unity to be adjusted with the week-day. Then in that context of increasing or decreasing the Ahargaṇa by a day the Adhimāsa-Sesha is to be increased or decreased by the number of Adhimāsās of a Kalpa and the Avama-Sesha is to be increased or decreased by the Avamās or Kshayāhās of a Kalpa.

Comm. After having computed the Ahargaṇa as per the directions given under verses 1–3 of the section of Grahānayana, we have to test its correctness on the basis of the week-day taking it that the Kalpa began with Sunday. In other words, since the number of the elapsed week-days accord with the number of Sun rises, the week-day on the day on which the Ahargaṇa is computed should accord with the Ahargaṇa ie dividing the Ahargaṇa by seven, the remainder must give us the week-day of the day in question. But it so happens that we may have to add or subtract one from the Ahargaṇa arrived at to adjust the Ahargaṇa with the week-day. Why does this happen?

The reason is as follows. When we find the Ahargaṇa for a particular number of elapsed tithis, we are unwittingly taking the number of elapsed true tithis in the place of the average tithis. The fact that we are given in the table of constants, the number of mean units of time in a Kalpa, for example mean solar days, mean tithis etc, and the fact that computation proceeds only on the basis of these mean units, means that we have taken into account only the number of mean tithis elapsed. Hence this is to be corrected by us. If we are computing the Al

for, say, the pratipat of phālguna, we automatically take that the elapsed true tithis from the beginning of the luni-Solar year is $11 \times 30 = 330$ tithis. But it may so happen that had we taken only the number of mean tithis elapsed, either 229 tithis or 331 tithis might have elapsed and not 330 as construed which therefore works an error amounting to unity in the Ahargaṇa arrived at. At the same time the error does not exceed unity, for, during the course of every lunation, the longer tithis are almost compensated by tithis of shorter duration in the same lunation. The variance in the length of a tithi being wrought by the variable motion of the Moon, which again depends on the distance of the Moon from his apogee, is rounded off in every lunation, as the Moon completes a circle with respect to the apogee in what is called an anomalistic month.

Thus it is that the Ahargaṇa is to be rectified on the basis of the week day. When this is done and the Ahargaṇa happens to be increased or decreased by unity, automatically it goes without saying, that the tithis are also increased or decreased by one. In this context there is a still more deeper significance as detailed below. The Adhimāsās are computed from the following formula viz. $\frac{A \times a}{S} = I + \frac{F}{S} \times 30$ (1) where A = Ahargaṇa, a = adhimasās in a Kalpa, S = solar days in a Kalpa, I = Integral quotient obtained by division and $\frac{F}{S} \times 30$ the Adhimasa-Sesha-tithis. If now A is to be increased or decreased by 1 $\frac{(A \pm 1)\,a}{S} = \frac{Aa}{S} \pm \frac{a}{s} = I + \frac{F}{S} \times 30 \pm a/S$ from (1) $= I + \frac{F \times 30 \pm a}{S}$; hence the adhimāsa-sesha namely $F \times 30$ is increased or decreased by a ie the number or adhimasās.

Similarly the Avama-Sesha or the Kshayāha-Sesha is to be increased or decreased by the Kshayāhas in a K...

The particular mention of the increase or decrease in the Adhimāsa-Sesha or the Avama-Sesha is necessitated in the context of computing the positions of the Sun or Moon, given the Adhimāsa Sesha and Avama-Sesha as mentioned under verses 6, 7 in the section of grahānayanā.

Verse 2. Pertaining to the smaller Ahargaṇa computed from the beginning of the current Solar year, called Laghu-Ahargaṇa.

In the case of computing the Ahargaṇa from the beginning of the Solar year also, when the Ahargaṇa is to be increased or decreased by unity, the tithis are to be increased or decreased by unity. The Avama-Sesha here is to be increased or decreased not by the Kshayāhās of the Kalpa but only by unity because we have used the formula $\frac{1}{64}$ of tithis to get the Kshayahās. If, an Adhikamāsa happens to occur during the course of the current year, the tithis, 30 in number of this Adhikamāsa must be also taken into account to obtain the Ahargaṇa.

Comm. Easy.

Verse 3, 4. The Ahargaṇa is to be computed (larger Ahargaṇa) after taking into account an Adhimāsa which has conspicuously occured but which is not obtained by computation or by rejecting an Adhikamāsa which has not occured but which is obtained by calculation. The Adhimāsa-Sesha is to be increased or decreased by the Adhimāsas of the Kalpa; the elapsed months from the beginning of the luni-Solar year are to be increased or decreased by unity and then the positions of the Sun and the Moon are to be computed from such an Adhimāsa-Sesha and such an Avama-Sesha.

Comm. Already explained.

Verse 5. A point to be noted with respect to the S'uddhi.

In the case of obtaining the S'uddhi, if an Adhikamāsa, which did not actually occur, is obtained by calculation, then the S'uddhi is to be increased by 30, so that the Ahargaṇa is not affected by the un-occuring Adhikamāsa.

Comm. The computation of the Adhikamāsas or intercalary months proceeds under the consideration of mean lengths. So, it is likely that an Adhikamāsa may occur un-warranted by calculation or may not occur in spite of its being shown by calculation. Further, an Adhikamāsa may be delayed in occurence by the fact that though the luni-Solar reckoning has gained over the solar by one mean lunation, the lunation at that point may still contain a Samkrānti, the preceding particular lunar month being smaller in length than the mean. Thus the convention made with respect to the occurence of an adhikamāsa, namely that the lunation which does not carry a Samkrānti is to be construed as an adhikamāsa, may also delay the occurence of the Adhikamāsa, though shown in calculation. Similarly an Adhikamāsa may be preponed though not warranted by computation by the same logic.

Verse 6. The criteria of an Adhikamāsa and a Kshayamāsa.

A lunation which does not carry a Samkrānti is an Adhikamāsa; whereas a lunation which carries two Samkrāntis is to be taken as a Kshayamāsa. The Kshayamāsa, occurs only in the course of the three lunar months named Kārtica, Mārgasīrsha and pausha and not during any other lunar month; when a Kshayamāsa occurs, then during the course of that year there will be two Adhikamāsās occuring on either side of the Kshaya-

Comm. The institution of intercalation has been explained to some extent under verse 10 of the Bhagaṇādh-

We shall see some more particulars of intercalation.

1. Bhaskara has given that 1593300000 Adhikamāsās occur in a Kalpa of 4320000000 Solar years, which means that 15933 Adhikamāsās occur in 43200 Solar years ie. 5311 Adhikamāsas in 14400 solar months. Converting 14400 into a continued fraction, we have $2 + \frac{1}{1+} \frac{1}{2+} \frac{1}{6+} \frac{1}{1+} \frac{1}{1+} \frac{1}{7+} \frac{1}{3+} \frac{1}{2}$. The successive convergents are $\frac{2}{1}, \frac{1}{3}, \frac{3}{8}, \frac{19}{7}, \frac{122}{55}, \frac{141}{52}$. Let us see what these convergents signify. (a) The convergent $\frac{19}{7}$ means that on an average there are 7 Adhikamāsās per 19 years. This ratio was adopted in the Romaka Siddhānta of Panchasiddhāntika. It means that $19 \times 12 = 228$ Solar months are equal to 235 lunations. The Metonic cycle described in modern astronomy is based upon this equivalence. The recurrence of Moon's phases in 19 years ie correspondence of the Moon's phases or tithis with the dates of the English year and Meton's formula are based on this equivalence. Recurrence of phase means recurrence of the relative positions of the Sun and the Moon, which again means recurrence of the S'uddhi, for S'uddhi is no other than the interval between the New Moon and Samkrānti. In the next verse Bhāskara says that a Kshayamāsa recurs after a lapse of either 19 years or 122 years or 141 years. The numerators of the last three convergents are 19, 122 and 141 which are the numbers of Solar years that effect recurrence of the same S'uddhi and as is going to be mentioned shortly a S'uddhi of 21 tithis is likely to bring in a Kshayamāsa. Hence a Kshayamāsa recurs either in 19 years or 121 years or 141 years.

2. Mention of the occurence of a Kshayamāsa was made by S'ripati first and not by the preceding astronomers. However, mention of it is there in the

literature and it is not clear whether observance of this Kshayamāsa was defunct for some centuries in between.

3. We shall now proceed to see how there occur two Adhikamāsas on either side of a Kshayamāsa. A simple argument is as follows. The convergents cited above namely $\frac{19}{7}$ or $\frac{122}{45}$ or $\frac{141}{52}$ signify that either 7 or 45 or 52 Adhikamāsās are to occur in the course of 19 or 122 or 141 Solar years normally. But when the S'uddhi happens to be 21 days at the beginning of the Solar year, it so happens that the S'uddhi goes on increasing for the first five months because the Sun is in his apogee when his longitude is 78°, and his motion being slow for three months when he is on either side of his apogee, the Moon gains over him in shorter intervals of time and as a consequence the lunations are of shorter duration. This means that the S'uddhi goes on increasing during those months and rapidly increases from its value 21 at the beginning to 30 by about Bhādrapada month. Under these circumstances, a Samkramaṇa occurs generally just before the beginning of Bhādrapada. The next Samkramaṇa happens just a little after the lapse of Bhādrapada, so that the month of Bhādrapada goes without a Samkramaṇa and as a consequence, it becomes an Adhikamāsa. Thus far it is alright that an Adhikamāsa has occured as per the meaning of the convergents. But when the Bhādrapada thus becomes an Adhikamāsa, the subsequent months from Kārtica to Mārgasira, being of longer duration than the corresponding Solar months, the Sun having a quicker motion on either side of his perigee, there is every likelihood of a Solar month being contained between two conjunctions or New Moon days. In other words two Samkramaṇas occur either in Kārtica or Mārgasira or Pausha. This means that as per the convention for the occurence of a Khayamāsa, one of the aforesaid lunations must become a Kshayamāsa. Thus the Adhikamāsa which

is due to occur during the course of the year, though it has occured has been lost. So, to make amends, another Adhikamāsa is to occur as is warranted by the convergents cited above. It might be asked what if two Adhikamāsās occur and why a Kshayamāsa be instituted at all. The reason is not that a religious convention warrants it but because the wedding of the luni-Solar year to the Solar year has to be made on a particular principle. Normally, so long as a Samkramaṇa goes on occuring during the course of a lunation, the two systems of reckoning may be seen to be running parallel. But if a particular lunar month does not contain a Samkramaṇa it is to be taken as a warning that the luni-Solar reckoning has overtaken the Solar by one lunation. This lunar month has to be curtailed to make the two kinds of reckoning to proceed side by side. This convention naturally raised the qnestion as to how to deal with a lunation which contains two Samkrāntis. The Solar month their has to be deleted to make the two systems run concurrently. This deletion of a Solar month is achieved not by declaring the particular Solar month as an Adhikamāsa but what is virtually the same two lunar months are deemed to lapse during the course of that solar month. This kind convention helped the occurrence of one Adhikamāsa alone as scheduled because one of the two Adhikamāsās has been nullified by the convention of a Kshayamāsa.

Why the proposition that a Kshayamāsa generally occurs when the S'uddhi at the beginning of the Solar year happens to be 21 tithis is quite evident because in such a case generally Bhādrapada becomes an Adhikamāsa, which again entails the occurence of two Samkramaṇas during the course of one of the three lunations beginning with Kārtica, which happen to be longer than the corresponding Solar months. In other words a Kshayamāsa is [illegible] to occur only when Bhādrapada happens to be an [illegible] māsa, and this in turns happens only when [illegible] to be 21 at the beginning of the Solar year. Adhi-

kamāsās do occur when the S'uddhi happens to be more than 21, but in this case one of the months prior to Bhādrapada happens to be an Adhikamāsa and then by the time Kārtica is reached, the Samkramaṇa occurs not at the beginning of Kārtica but a little later, which means that a second Samkramaṇa could not occur before the lapse of that lunation. This therefore will not be a Kshaya month.

The S'uddhi which is defined as the number of tithis in between a New-Moon and a Samkramaṇa, is clearly the interval which has been gained by the luni-Solar system over the Solar. This S'uddhi increases at the average rate of 11 days, 3 ghatis, 52 palas and 30 Vipalas per an year as we have seen already. For the occurrence of an Adhikamāsa during the course of an year the S'uddhi at the beginning of the year must be such that it accrues to a lunation during the course of the year. The apogee of the Sun being almost stationary having a very slow motion the lengths of the Solar months do not vary appreciably for a good number of years. Taking that the Sun's apogee is roughly at 80° longitude (78° according to Hindu Astronomy) the Sun's motion is less than his average from the moment when he has 350° longitude upto the moment when he has 170°, so that the Moon gains rapidly over him during this period. In other words the lunations during this period are of shorter duration i.e. the luni-Solar months of chaitra upto Sravana will be shorter in length so that S'uddhi increases rapidly from March upto August. Thereafter it will attain a stationary value just for a little time and then decreases for the next six months upto February. The increase, however, far exceeds the decrease and the balance of increase per year is as mentioned above is 11–3–52–30.

Bhāskara mentions in the course of the commentary that (1) the average length of a lunation is 29 days, 31 ghatis and 50 palas; (2) the average length of a Solar month is 30–26–17 (3) when the daily motion of the Sun happens to be 61′, then the Solar month will have a length

of 29-30 and as such falls short of a lunation and (4) that the minimum length of a Solar month is 29-20-40 only. Then he gives a general and broad explanation as to how a Kshayamāsa occurs on the following lines. Let us call NS as the interval between a New Moon and the subsequent Samkrānti. In other words it is the S'uddhi. Suppose Bhādrapada goes without a Samkrānti thus becoming an Adhikamāsa. Then in Āswayuja NS will have a small value, because the Samkrānti S which should have occured at the general rate of one per a lunation, has not occured before the lapse of Bhādrapada and being belated a little must have occured close on the heels of the New Moon of Bhādrapada. Let this interval NS have a particular value say x. Thereafter the Solar months grow gradually smaller i.e. the Samkrāntis occur earlier; so in Kārtica the value of NS decreases. It might decrease to such an extent that the next Samkrānti might occur before the next New Moon. Suppose this does not happen in Kārtica; then in Margasira the value of this NS is still smaller and there is a greater likelihood of another Samkrānti occuring during this Mārgāsira. Then the value of NS being the smallest in the month of Pausha, another Samkrānti is bound to take place during this month at the latest. In other words S now occurs before the next N i.e. the Samkrānti occurs just a little before the next New Moon. Let now this SN be a small quantity say y; or what is the same the next NS or the S'uddhi will be nearly a lunation. There after the Solar months begin to gain in length i.e. S occurs later and later. This means NS gains in length. Being nearly of a length equal to a lunation, and now gradually increasing it is bound to exceed a lunation shortly thereafter. This again means that S occurs later than the next New Moon, which is to say that one more lunation goes without a Samkrānti. Thus it is that another Adhikamāsa occurs shortly after the occurence of a Kshayamāsa. Hence in the course of one year there occur two Adhikamāsās and Kshayamāsa which means again that the balance of an

Adhikamāsa is there in that year. In other words, the occurence of a Kshayamāsa during the course of an year as per a stipulated convention does not preclude the average occurence of an Adhikamāsa which is normally due after an average lapse of $32\frac{1}{2}$ Solar months. That the occurence of a Kshayamāsa entails the occurrence of two Adhikamāsās on either side may be also seen in another way. Twelve lunations put together have a length of $354\frac{1}{3}$ days approximately during the course of which only eleven Samkrāntis could occur in case the initial S'uddhi is 21 days, for, subtracting this 21 from $354\frac{1}{3}$, there remain $333\frac{1}{2}$ days only which could contain only ten Solar months and not eleven. Ten Solar months are contained if there be only eleven Samkrāntis. Out of these eleven, two are consumed in a single lunation which happens to be a Kshayamāsa. Hence the remaining eleven lunations have to contain only nine Samkrāntis which means that two lunations have to go without Samkrāntis. In other words there are to be two Adhikamāsās during that year. Further both these Adhikamāsās could not occur on one side of the Kshayamāsa for the following reason. Since a Kshayamāsa could occur only during the course of Kārtica or Mārgasirsha or Pansha which alone are longer than the corresponding solar months, in the event of both the Adhikamāsās taking their place on one side of these months, they are to take place in nine lunations which are on one side of the three months beginning with Kārtica. This means that only seven Samkrāntis are to occur during the course of $9 \times 29\frac{1}{2}$ days $= 265\frac{1}{2}$ days. Even seven Solar months fall short of this period so that it is impossible that only seven Samkrāntis should occur in this period. Hence, the two Adhikamāsās which are to occur during the year containing a Kshayamāsa have to take their place on either side of the Kshayamāsa

Bhāskara mentions that when the daily motion of the Sun equals 61′, the length of the Solar month will be $\frac{1800}{61}$ $= 29$–30 approximately. He also states that the minimum

length of a Solar month will be 29-20-40. This will be so when the average daily motion during the course of a month is 61-20-30.

Verse 7. A mention of the years past and future that had and will have Kshayamāsās.

A Kshayamāsa occured in the Saka year 974 and will occur in the years 1115, 1256, 1378. Thus generally it occurs once in 141 years or even 19 years.

Comm. We have seen before that according to the convergents given above a Kshayamāsa generally occurs in 19 years or 122 years or 141 years. In the course of the commentary Bhāskara says "19 years before or after" so that 141−19=122 years was also meant by him. He gives how in the course of 141 years or 19 years the S'uddhi at the beginning of the year happens to be approximately 21 days. In this behalf, he invokes his formulation of the Adhikamāsās in the verse 6 under Pratyabdas'uddhi. As per that formula the number of Adhikamāsās in 19 years will be 7-13-37-30 taken by Bhāskara as 7-13-40. In other words the S'uddhi increases by 13 ghatis and 38 palas from what it was 19 years ago; thus practically the value of the S'uddhi recurs in periods of 19 years which means that if a Kshayamāsa occurred this year, there is a likelihood of its recurrence after nineteen years. Of course the recurrence would not be certain because there is an excess of 13 ghatis S'uddhi which might preclude its occurrence. Similarly in 122 years and 141 years the numbers of Adhikamāsās would be respectively 45 minus 7-15 ghatis and 52 plus 6 ghatis. 22 palas and 30 Vipalas, the latter being taken by Bhaskara to be 52 Adhikamāsās plus 6 ghatis and 20 palas. Thus Kshayamāsas are more and still more likely to recur in intervals of 122 years and 141 years respectively, the S'uddhis at the beginning of the year almost recurring and assuming the original value of 21 days for a Kshayamāsa to occur.

Bhāskara adds that a Kshayamāsa occured in 974 S'aka i.e. 62 years before his birth which knowledge he must have derived from hearsay or even by calculation. He then predicted its recurrence in the Saka years 1115, 1256 and 1378 which were at intervals of 141 years, 141 years and 122 years respectively. In the year 974 Saka i.e. 4153 Kali year, applying the above formula Dwidhābdāḥ the number of Adhikamāsās elapsed from the beginning of the Kali were 1531–21–12–52–30 i.e. the S'uddhi at the beginning of that year was 21–12–52–30. This naturally entailed the occurence of a Kshayamāsa. Thereafter during the years 1115, 1256 and 1378, the S'uddhis in the beginning of the years should be as per the above analysis 21–12–52–30 plus 68–22–30 i.e. 21–19–15; 21–25–37–30 and 21–25–37–30 and 21–18–23 respectively so that Bhāskara could forecast the occurrence of the Kshayamāsās in those years also. In this context it is worth-noting that Gaṇes'a notified in a verse that Kshayamāsās would recur in the years 1462, 1603, 1744, 1885, 2167, 2232, 2373, 2392, 2524, 2533, 2655, 2674,, 2796, 2815 according to Sūrya Siddhānta and according to Aryabhṭīya during the years 1482, 1793, 1904, 2129, 2186, 2251. These years also may be verified by computing the S'uddhis as said before and also with respect to Aryabhaṭīya according to which the number of Adhimāsas during a yuga differs by a little and as such effects a difference in the sequence of the Kshayamāsa years.

Verse 8. Tell me, how, when and in the course of how many years do two Adhikamāsas occur as mentioned by the Rishis? Questioned accordingly by an expert in questioning, if a mathematician could know the answer, I would reckon him as no other than Bhāskara (either the Sun-god or he himself i.e. Bhāskarāchārya) who could make the lotus-buds of mathematicians blossom.

Comm. Probably taking up this challenge alone Gaṇes'a answered the question and gave the years cited above which should bring in a recurrence of Kshayamāsās,

N.B. If does not suffice merely to compute the S'uddhis alone in the beginning of the years but a rigorous computation necessitates the calculation of the moments of New Moons and Samkrāntis also during the particular years as well, in as much as, the motion of the Moon also comes into the picture and it differs from month to month on account of the rapid motion of his apogee.

Here ends the section named Adhimāsādi-nirṇaya.

THE SECTION BHŪ-PARIDHI-MĀNĀDIKA- CIRCUMFERENCE OF THE EARTH

Verse 1. The circumference of the earth's globe is 4967 yojanas; its diameter 1581. A yojana is equal to $\frac{d \times 360}{7}$ where $\delta\phi$ is the difference in the latitudes of two places on the same terrestrial meridian in degrees, C the circumference of the earth's globe given above and *d* the distance between the two places.

Comm. The second half of the verse gives the method of computing the circumference of the earth's globe, which is mathematically correct; for, by the rule of three "If by a difference of 90° in latitude we have $\frac{1}{4}$ C, what shall we have for 1° difference in latitude?" The answer is $\frac{C}{360}$. Again "If by distance of d between two places on the same meridian, we have a difference of $\delta\phi°$ in latitude, what shall we have for 1° difference in latitude?' The answer is $\frac{d}{\delta\phi}$. Both the answers must be the same; so equating them, we have $\frac{C}{360} = \frac{d}{\delta\phi}$ = number of yojanas = x say. Hence one yojana $= \frac{x}{x} = \frac{d}{\delta\phi} \div \frac{C}{360} = \frac{d \times 360}{C \times \delta\phi}$

N.B. Here it must be noted that a yojana's length is derived from the number of yojanas contained in the circumference as reported in the Āgama.

In the course of the commentary under this verse, Bhāskara explains why he had recourse to this kind of definition, which is based upon āgama and as such does not contain a proof. He says that in as much as the defi-

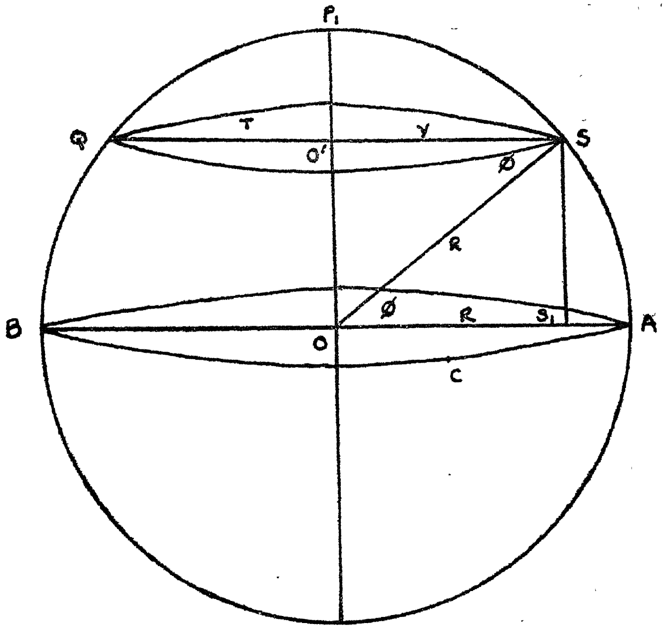
nition of a yojana was given basing ultimately on the units of Angulas and yavas (grains of paddy) differently by different authorities, and in as much as the circumference of the earth's globe was more or less unanimously accepted, he has chosen to define a yojana on the accepted measure of the circumference. In some other place Bhaskara says that a yojana is equal to 4 Krōsās, where the word Krōsa etymologically means that distance through which the topmost voice of a healthy person could be heard by another healthy person having good audition.

In this context it may be recalled how S'ripati defined a yojana under verses 69, 70 Madhyamādhyāya of his Siddhānta S'ekhara. "The minute speck of dust that is seen flying in the light of the Sun's rays entering a house through the windows, is called a paramāṇu (not an atom as is being translated now). Eight such paramāṇus equal one Reṇu. Eight Reṇus equal the breadth of the end of a hair known as Vālāgra or Kaccha-mukha or yūka. Eight Vālāgras equal one Likshā; Eight Likshās equal one Yava; eight Yavas equal one Angulā; twelve Angulās make what is called a Vitasti (i.e. the expanded length of a palm of the tallest person). Two Vitastis equal one Hāsta. Four hastās make one Chāpa. Two thousand Chāpas make one Krōs'a and four Krōs'ās are reported to be equal to a yojana by Astronomers"

Verse 2. Rectification of the circumference .ie. finding the length of the circumference of the earth parallel to the terrestrial equator and passing through the locality in question. Also the definition of the primary meridian.

The equatorial circumference of the earth multiplied by cos ϕ and divided by R or multiplied by 12 and divided by the hypotenuse of the right angled triangle formed by the gnomon and the equinoctial midday shadow thereof. (Hereafter called equinoctial hypotenuse) gives the circumference of the earth parallel to the equator and passing

through the locality (hereafter called the rectified ference). Also the primary terrestrial meridian is that longitudinal line passing through the places (1) Lanka (2) Ujjain (3) Kurukshetra and (4) the north pole.



Comm. (Ref. fig. 4) Let ABC be the terrestrial Equator, and QST be a circle parallel to this terrestrial equator and passing through the locality S. This QST is called the rectified circumference of the earth at the place S. The terrestrial equator is defined as follows. It is known that the earth rotates about herself about the axis $P_1 P_2$ where $P_1 P_2$ is called the polar axis or Dhruvayashti. In other words the entire heavens appear to revolve round the earth in such a way that any star will appear to be revolving in a circle called its diurnal circle which is parallel to circle ABC. The points P_1 and P_2 are called the north

south pole respectively. These two points evidently do not move, though the earth is herself rotating in the clockwise direction. If O be the centre of the earth, OP_1 produced meets the skies at a point called the north celestial pole and OP_2 produced meets the skies in the point called the south pole. It so happens that the north celestial pole is very near a star which is called the pole star. This star is known as the Dhruva-Tāra in as much as it does not appear to move at all while all the heavens (i.e. all the stars of the sky) appear to be revolving round the earth rising and setting as seen at any place. (The word Dhruva means fixed). Aryabhaṭāchārya mentioned in so many words that it is the earth that really rotates and so the stars which are themselves fixed appear to be going round the earth in circles parallel to the circle ABC. अनलो

f

॥ (Explained under verse 7 Bhagaṇādhyāya). The celestial Equator is the great circle which is the circle of intersection of a plane perpendicular to the polar axis and passing through the earth's centre with the celestial sphere (celestial sphere is the sphere-like surface which shape the Sky takes and on which the stars and the planets appear to be studded.) Similarly the terrestrial equator ABC is the circle of intersection of the earth's globe with the same plane. Thus the terrestrial and the celestial equators are concentric coplanar circles with the earth's centre as the common centre. A great circle of a sphere is a circle whose plane passes through its centre. Thus ABC is a great circle on the earth's surface, because its plane passes through the earth's centre. Similarly the celestial equator is a great circle of the skies. The circle QST is called a small circle, just as the diurnal circle traced by any star in its diurnal rotation is also a small circle parallel to the celestial equator. Thus small circles, an infinity of them can be drawn parallel to the terrestrial equator ABC and they will be in decreasing dimension as we proceed towards the pole. Hence the Sphutaparidhi or the rectified circumference

QST at the locality S is a small circle whose circumference is smaller than that of ABC. The problem is now to find the length of this circle QST. Evidently $\frac{QST}{ABC} = \frac{2\pi r}{2\pi R}$ $= \frac{r}{D}$ where r and R are respectively the radii of the two circles. But $\frac{r}{R}$ = Cosϕ from the triangle OSO′, where

$O'_1SO = SOA$ = latitude of the place S. QST =

$$ABC \times Cos\phi = \frac{ABC \times H\ Cos\ \phi}{R} \quad I$$

the Hindu cosine of ϕ known as lambajyā and stands for $OS_1 = O'_1S = r$. This lambajyā is also called Dyujya if the small circle is the diurnal circle of a Star. The diurnal circle of a Star is called Dyujya-Vritta and its radius is called Dyujyā-Equation I proves the first statement of the

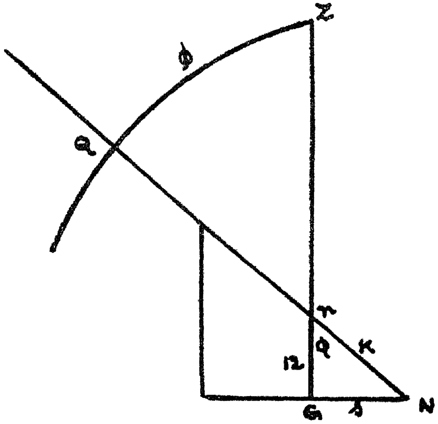


Fig. 5

verse. In fig. 5 Gn N is called the fundamental gnomonic triangle where Gn is the vertical gnomon pointing to the Zenith Z of the celestial sphere and is considered to be of 12 units (Angulas as they are called), GN the midday-shadow of the gnomon cast on an equinoctial day when the

Sun is at the point Q where Q is the point of intersection of the celestial equator with the meridian of the place. If the equinoctial shadow be denoted by 's' and the hypotenuse of the triangle nGN namely nN be denoted by K (called the Vishuvatkarṇa or the equinoctial hypotenuse) then $\frac{s}{K} = \text{Sin}\,\phi$ and $\frac{12}{K} = \text{Cos}\phi$. If H Sin ϕ be the Hindu sine of ϕ called the Akshajyā H Sin ϕ = R Sin ϕ so that $\frac{s}{K} = \frac{\text{H Sin}\,\phi}{R}$ and $\frac{12}{K} = \frac{\text{HCos}\phi}{R} = \frac{\text{Lambajyā}}{\text{Trijyā}}$.

Substituting for $\frac{\text{H Cos}\,\phi}{R}$ the value $\frac{12}{K}$ in equation I above, we have

$$QST = \text{Sphuta - paridhi} = \frac{\text{Bhū - paridhi} \times 12}{K} \qquad \text{II}$$

which proves the second statement given in the verse.

Regarding the third statement, which defines the Bhu-Madhya-Rekhā (In modern text books of geography the terrestrial equator is spoken of as the Bhu-Madhya-Rekha. The terrestrial equator is called Niraksha-Rekha in Hindu Astronomy which means the circle of Zero-latitude), it is the primary meridian taken by Hindu Astronomers. In modern astronomy the primary meriodian is taken as the Greenwich meridian. Bhāskara has given four places located on the Hindu primary meridian, but, S'ripati gives many more places located on this primary meridian under verse 96 Madhyamādhyāya namely (1) Lankā (2) Kanyākumāri (3) Kānchī (4) Pannāta (5) the six-faced white mountain (6) Sri-Valma-gulmam (7) Māhishmatī (8) Ujjain (9) An Āsrama (10) Pattasiva, a town (11) Sri Gargarāna (12) Sthānvīswara known also as purohita (13) S'ītagiri and (14) Sumeru. Some of these places cannot be properly identified but the following remarks may be made (a) Pannāta is one of the fifty-six small countries into which India was divided in ancient times according to the purānic literature

(b) It is not clear what places are indicated by (5) and (6) cited above (c) In some works Māhishmatī and Ujjain are used synonymously. (8) is not clear. Regarding (9) there is one pattas'iva near Rajahmundry but S'ri pati does not seem to have meant it. Again (10) is not clear, Regarding (11) it is to be noted that the place is now pronounced (probably mis-pronounced) as Sthānes'wara. If (12) means the Himalaya mountain, there is not much meaning to say that it lies on the primary meridian; only a cross-section of it could lie there upon. The entire mountain extends from west to East over more than a thousand miles.

Verse 3. To find the correction known as Des'āntara.

The distance between two places on the same latitude multiplied by the daily motion of a planet and divided by the rectified circumference is a correction subtractive in the east and additive in the west of the primary meridian in the planetary position obtained.

Comm. In Hindu Astronomy the mean planetary positions are first calculated for the Sun-rise at the primary meridian. Now suppose a place lies to the east of this meridian. Then the Sun-rise at the place happens to occur earlier than on the primary meridian. Hence the correction in the mean computed position of the planet is negative if the position were to be calculated for the local Sun-rise. If the place happens to be on the western side of the primary meridian the reverse holds good i.e. the correction is to be additive. The amount of the correction is the amount of the motion of the planet in between the two Sun-rises. Let the planet move an arc equal to δm per day i.e. it moves δm when the earth rotates once about her axis. The time between the two Sun-rises above is the time by which the local meridian is carried through the distance between the locality and the primary meridian's point of inter-section with the latitudinal line or what is the same through the arc of the rectified circumfer-

ence of the earth pertaining to the locality. If d be this distance then the rule of three to be used is "If the length of the rectified circumference viz. C rotates by the time the planet moves a distance δm, what is the arc traversed through by the planet if an arc 'd' of C rotates through?" The answer is $\frac{d \times \delta u}{C}$ which is the correction required.

Verses 4, 5, 6. The Correction Des'āntara expressed in time.

The eclipse of the Moon occurs at a place situated on the east of the primary meridian later than on the primary meridian and vice versa. The time in between the two moments is the Des'āntara expressed in time. The distance of Des'āntara ie. the distance of the locality from the primary meridian measured along a parallel to the terrestrial equator or Niraksha Rekhā is obtained by multiplying the rectified circumference by the Des'āntara measured as above in ghatis and dividing by 60. Also the above time in ghatis multiplied by the planets' daily motion and divided by 60, gives the correction in arc in the computed mean planetary motion.

Further the week-day begins after or before the local Sun-rise by that Des'āntara expressed in time according as the locality is on the east or west of the primary meridian. Also the week-day begins after or before the local Sunrise by the ghatis of the correction known as chara according as the Sun is in the northern or Southern hemisphere.

Comm. An eclipse is first computed for the primary meridian. If an observer wants to know whether he lies east or west of the primary meridian and to know the Des'āntara correction in time, the following procedure is to be adopted. Let a lunar eclipse begin x ghatis after the Sun-rise of the primary meridian. Let the observer note the time y ghatis which have elapsed after Sun-rise at his own place when the eclipse begins. Since a lunar

begins simultaneously for any place of the earth, if $y > x$, then he should know that he lies on the east of the primary meridian because his Sun-rise happens to be earlier than the Sun-rise on the primary meridian. Also the difference $y-x$ gives the Des'āntara correction in time for his place. The converse is the case if he happens to lie on the western side of the primary meridian. The time at which the eclipse takes place on the primary meridian after the Sun-rise there which is obtained by computation is called Drik-grahaṇa - Kāla; whereas the local time after Sun-rise observed by the observer is called pragrahaṇa-Kāla. Their difference is therefore the Des'āntara correction in time.

If the Des'āntara is to be got in yojanas, $\frac{T \times C}{60}$ is the answer, where $T = y - x$ and C is the rectified circumference of the earth, for, if a difference of 60 ghaṭis be there for C yojanas, what should be the distance in yojanas in order that the difference is T''? The answer is as given above.

Hence to obtain the positions of the Sun and the Moon at the beginning of the eclipse at the locality we have to add or subsract as the case may be $\frac{T \times \delta m}{60}$ where δm is the daily motion of the Sun or the Moon, and T is $y-x$ cited above, for, "If in 60 ghaṭis the motion be δm, what would it be in T?" is the rule of three for which the answer is as stated above.

Now the question is when the week-day begins for the locality. It must be noted clearly, that in Hindu Astronomy the moment of Sun-rise at the primary meridian alone is to be reckoned as the beginning of the week-day universally. This convention is adopted for convenience. Thus the astronomical week day for any locality does not begin from the Sun-rise of the locality, but may begin earlier or later. This difference is given by $y - x$ cited above.

There is yet another subtlety in the commencement of the week-day, arising out of the latitude of the place. The former analysis pertains to the longitudinal difference. The difference arising out of latitude between the local Sun-rise and the Lanka-Sun-rise is given by what is called Chara-Kāla. Since the week-day begins at Lanka Sun-rise and the local Sun-rise differs from the Lanka Sun-rise not merely by a longitudinal difference but also by a latitudinal difference, to compute the actual beginning of the week-day before or after the local Sun-rise, we have to take into account both the differences cited above. In other words, computing the local Sun-rise and also the Lanka Sun-rise, we have to decide the beginning of the week-day before or after the local Sun-rise.

Verses 7, 8. The correction called Bījakarma for the planetary positions.

The number of years from the beginning of the Kalpa divided by 12000, the remainder, or the difference of the divisor and the remainder whichever is less is to be divided by 200. The quotient in minutes of arc, multiplied by 3, 5, 5, 15, 2 respectively is a negative correction in the positions of the Sun, Moon, Jupiter, Venus, and the lunar apogee and multiplied by 1, 52, 2 and 4 gives the positive correction in the positions of Mars, Mercury, the lunar Node and the Saturn respectively.

Comm. By the phrase 'The remainder or the difference of the remainder and the divisor', it is plain that the corrections positive or negative increase for 6000 years and decrease for the next 6000 years. Bhāskara gives no reason for these corrections, but, we have to construe these corrections on the following rational grounds. Bhāskara, however, says that the corrections were accepted by him on the basis of Āgama. This Āgama—stipulation was there in Brahma–Sphuta–Siddhānta and was later incorporated by S'ripati also in his Siddhānta–S'ekhara and as such was

accepted by Bhāskara also. However, in Brahma Sphuta Siddhānta both as first published as an edition of M. M. Sudhākara Dwivedi and later by the late Rāmaswarupa S'arma in 1966, the verses 59, 60 of Madhyamādhikāra suggest that the corrections are negative in the case of all the planets; whereas both S'ripati and Bhāskara make them positive in the case of the latter four viz. Mars, Mercury, the lunar Node and Saturn. By this we have to construe that S'ripati and Bhāskara must have had before them a text which should have read 'स्व' in the place of 'च' in the last pāda of verse 61. M. M. Sudhākara-Dwivedi did not notice this anomaly of the positiveness of the correction with respect to the latter four, but he remarked, however, that there was a prosodial lapse in the last pāda of verse 61, for which he offered a suggestion that instead of वेद, we had better read वेदै:—This suggestion, no doubt, rectifies the prosody of the verse, but not the the anomaly cited above which was not noticed by M. M. Sudhākara Dwivedi. So, we have offered our own suggestion namely that in the place of च as mentioned above if we read स्वं, we not only rectify the prosodial error but also the anomaly referred to. Rāmaswarupa S'arma noted the anomaly but did neither refer to the prosodial error nor offer a correction. It seems that Rāmaswarupa S'arma did not verify the corrections stipulated from the verses 91, 92, 93 of Madhyamādhyāya of Siddhānta S'ekhara. In this latter work, there is another anomaly namely that in the case of Mercury, the number 62 is the multiplier and not 52. Makkibhatta, the ancient commentator had before him a text which read 62 in the place of 52, in all probability, a mistake of the scribe. M. M. Sudhākara Dwivedi is reported to have later pronounced that 52 must be the correct figure when this was brought to his notice as this number 52 was found both in Brahmagupta and Bhāskara. As reported by the editor of Siddhānta S'ekhara Pandit Babuaji Mishra, who mentions this latter pronouncement of Sudhākara Dwivedi his teacher, also says that Sudhākara Dwi-

vedi suggested the reading ह्रिशर in the place of ह्रिरस of verse 93 of Siddhānta S'ekhara. The fact that Makkibhatta commented ह्रिरसलङ्गुणं as ह्रिषह्रिसलङ्गुणं shows that he did not consult Brahmasphuta Siddhānta in this place; also, he must have had a manuscript before him which scribed ह्रिरस in the place of ह्रिशर. Using श ष, स, indiscretely is not uncommon in many books of North India, from a long time and the scribe of the manuscript probably having used स in the place of श and then by an oversight a latter scribe having inverted सर as रस, Makkibhatta must have commented like that.

Incidentally a remark may be made here about Makkibhatta. He was evidently a keralite because he used letters to signify numbers as was a common practice among the Kerala Astronomers, and as he also commented upon Bribad-Bhaskarīya. Further, it is interesting to note that he wrote in his commentary under verse 39 of the Sādhanādhyāya of Siddhānta S'ekhara viz. "भभ्रमोऽत्र नरमण्डलान्तरं सावनानि कुदिनानि तानिवा", "भूमेः प्राङ्मुखी भ्रमति" etc". This idea shows that he accepted Aryabhata's verse "अनुलोम गतिः etc" implying that the earth is rotating.

Bhāskara says that the Bīja correction mentioned was purely based on Āgama and Upalabdhi (meaning authority and observation'). M. M. Sudhākara Dwivedi seems to have reiterated the same as reported by Babuaji Mishra, in a foot-note. Kamalākara, condemned this Bijakarma as it was unwarranted and had no proof.

A rational explanation as to why this Bīja-Karma was prescribed either by Brahmagupta himself or some authority which he seems to have accepted may be given as follows. The small differences in the numbers of sidereal revolutions or what is the same the minute differences in the accepted daily motions of the planets and the assumption of a conjunction of all the planets and planetary points at the beginning of Kalpa, which is beyond proof,

resulted in a difference between the computed planetary positions and their observed positions. So, the originator of this Bija-Samskāra, noting the differences in his own time devised a formula, which could account for those differences. But this formulation was bound to go wrong in later times as long as the daily motions are not corrected to the minutest extent possible and as long as the fundamental basis of the conjunction of all the planets and planetary points is not proved. This seems to be the reason why so may texts were written incorporating small differences in different times as reported by Gaṇēśa (1507 A.D.) in his work Brihat-Tithi-Chintāmaṇi in the words " The calculations of planetary positions according to the methods indicated by Brahma, Vasishtha and Kasyapa Siddhantas held good in their own times, but grew obsolete later; Then Maya, the demon at the end of Krita obtained the science from the Sun God, which again grew obsolete in this Kaliyuga wherein parāśara began to hold the ground for a good length of time. Then Āryabhata rectified the methods; when even those methods grew obsolete, Durga-Simha, Varāha Mihira and others set them right. Again Brahmagupta came into the picture to rectify the methods by his own observations. Then Came Kēśava (Gañeśa's father) who rectified further. After a lapse of sixty years, his son Gaṇeśa has now to correct the Science. If this also grows obsolete (as it is bound to) in course of time, let others again rectify it by observing conjunctions of the Moon and planets with the asterisms."

Obsoleteness arises out of two contexts, one a justifiable situation and the other based upon a wrong premise. The first is as follows. Suppose as a first approximation we take the length of an year as 365 days. We will have committed an error nearly $\frac{1}{4}$ of a day, so that the error accures to a day in 4 years. Thus the convention of the leap year arose so that during four years we give a day more to February. Here again we have overestimated the error by nearly $\frac{1}{100}$th of a day. Hence in 400 years the

above correction leads to an error of a day. So, it is that we pronounced that out of the years 2000, 2100, 2200, 2300 A.D., the year 2000 A.D. alone is a leap year and not the remaining, the convention being that the number of the century, here 20, must be also a multiple of four. On this back-ground, suppose we prepare a manual called a Karaṇa grantha taking the length of the year to be 365·25 days. It works alright for some time but in the course of 400 years the error will have reached to as much as one day. Thus a manual like the above works only for a short time and the approximation made gradually brings in a divergence on account of which such a manual grows obsolete. That is why one Narasimha who happened to prepare a manual in 1333 Saka year (1411AD) opens his work with the words "तिथिचक्रं यत्प्रणीतं मल्लिकार्जुनसूरिणा, कालेन महता तस्मिन् खिलीभूते तदादरात्, नौपुरीसिङ्गयार्यस्य नरसिंहेनसूनुना एतदेव स्फुटतरं क्रियते सौरसम्मतम्" i.e. "In as much as a manual named Tithi-Cakra prepared by one Mallikārjuna Sūri long ago, based upon the Sūryasiddhānta has now diverged far from the Sūryasiddhānta (on account of the approximations made transcending the limits of negligibility) I, the son of one Singaya belonging to a place named Nau-puri (probably Vada-palle of the East Godavary Dt.) am rectifying it and bringing it to accord with the Sūrya Siddhānta again."

This kind of obsoleteness arising out of inevitable approximations that have to be made in the preparation of manuals is permissible. But Suppose the premise of the manuals itself is incorrect, then the rectification of the manuals is no good so long as the data given in the premise are not corrected. There are two fundamental defects in the ancient works according to a modern analysis namely (1) The Supposition that all the planets were in conjunction at the Zero-point of the Zodiac in the beginning of a Mahāyuga (2) Small variations in the constants like the daily motion of the planets and the like. According to the modern interpreters of Hindu Astronomy the

first premise was not correct. According to them, some astronomers having observed the daily motions of the planets or what is the same the sidereal periods of the planets to a sufficiently good approximation calculated back or extrapolated a date on which these planets should have been in conjunction at the Zero-point of the Zodiac. The extra-polated date was naturally wrong to some extent because the sidereal periods found could not but be correct only to a particular degree of approximation. Thus a little alteration in the number of sidereal revolutions alone or the number of days in a Mahayuga made to suit the observed positions at a particular epoch would be only a tinkering of the problem and not a true solution. Thus Hindu Astronomy could be saved and its methods could still be followed provided instead of trying to presume a date at which all the planets were in conjunction (No doubt in the long bosom of time, such a presumption also could not be ruled out) correctly observed positions of the planets by the help of modern instruments were taken as the basis of an epoch and thereafter using more correct values of the constants such as the sidereal revolutions, maximum equations of centre and maximum S'ighraphala, obliquity of the ecliptic etc. The second defect cited above thus being removed, and the original premise being changed, the methods of calculation still hold good and there would be no necessity to be going on with tinkerings of the problem.

The Bīja-correction which we are commenting upon was rightly criticised by Kamalākara as irrational though he himself fanatically tried to uphold Surya Siddhānta. Even today there are a good number of the traditional Hindu Astronomers who do hold that the Sūrya Siddhānta was revealed to Maya at the end of Kritayuga in spite of the fact that scholars like M. M. Sudhākara Dwivedi pronounced that it was an extra-polated work shortly after the time of Brahma-Guptāchārya. It is interesting to note that Bhāskara, a very rational astronomer, had before him the verse "त्रिंशत्कृत्वो युगे भानां चक्रं प्राक् परिवर्तते" of the

Sūrya Siddhānta (verse 9 ch. 3). He did not give it the interpretation that was later put upon it through the two subsequent lines "तद्दोः त्रिघ्ना" etc., which lines were not there evidently in Bhāskara's time. Without these two latter lines the rate of precession was too small to be accepted by Bhāskara and so he chose to follow Munjāla rather than the Sūrya Siddhānta. Our Traditional astronomers today have no reservation to accept the greatness of Bhāskara and worship him though they do not question what necessity Bhāskara had to write another treatise and that too basing it upon the Āgama accepted by Brahmagupta and not Sūrya Siddhānta, when there existed Sūrya Siddhānta before him and from which he had no objection to quote verses like "अहश्यरूपः कालस्य मूर्तयः" etc. (verse I ch. 2.)

The Bīja-correction first incorporated by Brahmagupta and later followed by a good number of astronomers because S'ripati and Bhāskara accepted it, will not be acceptable to modern astronomers, though it might have worked well at the time of Brahmagupta and for some years later. The reason is that it is construed only as a tinkering of the defect as explained before.

It is also to be noted that the originator of this Bīja-correction did not make it secular i.e. valid for all time increasing without a limit, for, then, the respective corrections transcend all limits and render the corrections meaningless. So, he said that the corrections would be increasing for 6000 years and thereafter begin to decrease to nothing. They were Zero at the beginning of the Kali because all the yugas are multiples of 12000 years. Also the maximum correction is in the case of Mercury $\frac{6000}{200} \times 52 = 1560' = 26°$. Let us see how far this is justifiable. The daily mean motion of Mercury as given by Bhāskara is $4° - 5' - 32'' - 18''' - 28''''$ whereas as per modern astronomy it is $4° - 5' - 37\frac{7}{8}''$ approxly. So there is a positive error of $5''\frac{23}{40}$ which will accrue to $18' - 35''$ in

course of 200 years. But as per the Bīja-correction it should be 52'. Hence it is a fact that there is a positive error but not so much as indicated. But it must be noted that Mercury's orbit has the highest eccentricity of as much as .2, and the observer who stipulated the correction must have observed when Mercury was near its perihelion, where the error could have been as much as indicated and even more. Similarly on close analysis it could be proved that the Bīja-correction should have been as indicated, say, roughly about 3300 Kali era, which might be roughly the date of its stipulation.

Verses 9, 10. Concluding verses of the Madhyādhikāra.

If the work is made more voluminous by describing various methods which are easy and interesting to unintelligent people, learned men look down upon such a work as ~~indulging in~~ unnecessary verbosity. Hence the volume of a work does not add to its greatness; So I have made my work neither voluminous nor brief-worded. The reason is that both the intelligent as well as the unintelligent people are to be enlightened.

For the sake of clarity of exposition, different ingenious methods being used in such a way that the work does not exceed the normal limits of the previous works, and incorporating as far as possible unit numerators, fractions having numerators and denominators mutually prime, using methods of interpolation and reduction, making use of different kinds of denominators and numerators in many ways, this kind of treatment must be given to a work of this nature by an intelligent man.

Comm. Easy.

Before we proceed to the next chapter, we shall add here tables of astronomical constants as given by different authorities, which will help comparison and appreciation of the work.

TABLE 1

Number of sidereal revolutions of planets and planetary points in a Kalpa

	Modern Sūrya Siddhānta	Bhāskara	Āryabhata	Khaṇḍa Khādyaka	Mahā-siddhānta
Sun	4320000000	4320000000	4320000000	4320000000	4320000000
Moon	57753336000	57753300000	57753336000	57753336999	57753334000
Mars	2296832000	2296828522	2296824000	2296824000	2296831000
Mercury	17937060000	17936998984	1793702000	17937000000	17937054671
Jupiter	364220000	364226455	364224000	364220000	364219682
Venus	7022376000	7022389492	7022338000	7022388000	7022371432
Saturn	146568000	146567298	146564000	146564000	146569000
Moon's Apogee	488203000	488205858	488219000	488219000	488208674
Moon's Node	232238000	232311168	232226000	232226000	232313354

TABLE 2

	According to Sūrya Siddhānta	According to Bhāskara
(a) Number of mean solar days in a Kalpa	157791728000	1577916450000
(b) Number of Adhikamāsas in a Kalpa	1593336000	1593300000
(c) Number of Kshayāhas in a Kalpa	25082252000	25082550000
(d) Diurnal revolutions of stars	1582237828000	1582236450000
(e) Tithis in a Kalpa	1603000080000	1602999000000

TABLE 3

Daily mean motions of planets

	Brahmagupta S'ripati and Bhāskara
Sun	$0-59'-8''-10'''-21''''$
Moon	$13°-10-34-53-0$
Mars	0–31–26–28–7
Mercury	$4°5'-32''-18-28$
Jupiter	0–4–59–9–9
Venus	1–36–7–44–35
Saturn	0–2–0–22–51
Moon's Apogee	0–6–40–53–56
Moon's Node	0–3–10–48–20

SPAṢṬĀDHIKĀRA — RECTIFICATION OF PLANETS

Introduction. In the Bhagaṇādhyāya section of the previous chapter Bhāskara gave under the Caption Bhagaṇopapatti his proofs as to how the ancient is might have obtained the number of sidereal revolutions of the planets and the planetary points called apogees or aphelia and Nodes. But in trying to give those proofs, he was aware and he confessed also in so many words that some of his proofs at least were obsessed by what is called Itaretara-rāśraya-Doṣa i.e. "answer begging the question" It is worth-hearing his words in his commentary under verses 1–6 of the section cited above—" That the planets, and the planetary points perform so many revolutions in a Kalpa, is essentially conveyed by the Āgama i.e. the Śāstra (which is to be taken on faith). That Āgama, got diversified i.e. there are many versions of that Science, due to the defects of scribes, the teachers and the students and due to a long lapse of time from the originators of the Āgama. That being so, the question arises as to which of the versions is to be trusted as the right authority. If it be said so, in mathematics only an āgama which could be proved also should be taken as authority. Such a number of revolutions as are obtained by proof, is to be accepted. Even that could not be (a proof) ; for, a great scholar could just understand the proof and by that proof alone, it is not possible to know the exact number of revolutions (in a kalpa), for, a man's longevity is not much. In the proof that could possibly be given, the planet's position is to be observed and noted every day, during the entire course of its revolution. Thus Saturn Completes its sidereal revolution in about 30 years. The apogee of the Sun and the aphelia of the planets have their revolutions running into hundreds of years. Hence the observation of one complete revolution (of such a planetary point) is beyond the capa-

city of a mortal. Hence great astronomers accept such an āgama as would give results which accord with observations during their times, and such a one as was formerly accepted by a very intelligent astronomer. Then they produce their own works exhibiting their own Skill in the Science and refuting wrong notions of others. Their idea is 'Let the Āgama we take as an authority be whatever it would be Let us show our own skill in the course of our work-', just as in this work, the āgama accepted by Brahmagupta is taken on faith as the authority. Then it might be argued "Better not attempt at trying to prove how the numbers of sidereal revolutions were arrived at. Even if a proof be attempted, that proof *would be obsessed by the* '*Itaretarā-śrayadōṣa*' (cited above). Nevertheless we shall give a brief proof, That 'itaretarāśrayadōṣa' is apparently a dōṣa i.e. an apparent defect; for, different proofs could not be adduced simultaneously. The proof will now be given".

These words indicate that even such a highly rational and supremely intelligent astronomer like Bhāskara could not set aside his faith in our āgama and attempt at a pure and rigorous proof, which would not invoke the āgama-Let us see where in his proof he does commit the so-called itaretarāśrayadōṣa and where he invokes the āgama. Also we shall try to construct a proof, of course to a good extent on the lines on which Bhāskara tries to give his proof, but at the same we shall not invoke the āgama. where he does, but try to proceed purely on a rational basis. We shall take up the proof under verse 18, in its appropriate context. We shall now proceed with the text upto that point, which gives a brief sketch of the Hindu trigonometry.

Verse 1. In as much as true positions of the planets alone are required to decide auspicious moment; for journeys, marriages, celebrations pertaining to temples, astrology and the like, we shall now give the methods of rectifying the mean positions of the planets so as to accord with their observed positions.

Comm. Clear.

Verse 2-9. Obtaining the sines of the angles and tabulation of the sines.

The planet deflected to the true position from the mean lies at the end of a half-chord (which is the Hindu sine of an angle) so that many processes pertaining to a planet are carried through Sines of angles; hence the word half-chord alone is connoted in this work wherever the word Jyā meaning a chord is used.

The lengths of these half-chords (or the Hindu Sines) for angles increasing from 0° to 90° at intervals of $3\frac{3}{4}°$ are as follows—225′, 449′, 671, 890, 1105, 1315, 1520, 1719, 1910, 2093, 2267, 2431, 2585, 2728, 2859, 2977, 3084, 3177, 3256, 3321, 3372, 3409, 3431, 3438. The ut-kramajyāis or the Hindu versed-sines are respectively 7, 29, 66, 117, 182, 261, 354, 461, 579, 710, 853, 1007, 1171, 1345, 1528, 1719, 1918, 2123, 2333, 2548, 2767, 2989, 3213, 3438.

The word Tribhajyā or Trijyā is half-diameter. The word jyā khandas used by pandits connote the differences between successive sines.

Comm. There is a difference between modern trigonometrical sines and the Hindu sines as detailed below (Ref.

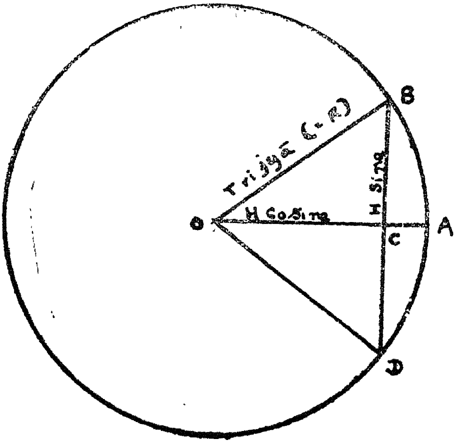


Fig. 6

fig. 6 overleaf). Let (O) be a circle i.e. a circle with centre 'O'. Let AB be an arc called 'Chāpa'; let BC be drawn perpendicular on OA; then BC is half of the full chord BD (known as jyā). The half-chord Ardha-jyā is itself spoken of as jyā for convenience and is the Hindu-sine of the arc or chāpa AB. In Hindu trigonometry 'angle' is connoted by the arc corresponding to it and as such spoken of as chāpa. OC is spoken of as the Hindu-cosine or Koti-jyā and CA is called the ut-kramajya or the Versed-sine. The radius O B is called trijyā and let us connote it by R. To differentiate between the modern terms and the Hindu terms, we use the words H. Sine, H. Cosine, H. vers-sine for the Hindu sine, the Hindu cosine and the Hindu vers-sine respectively. Also the radius R is generally taken to be 3438′ which, we know to be the approximately the minutes in a radian. To talk of a length in minutes appears rather odd but no confusion need be there, for, an arc of length R subtends 3438′ at the centre. It is called Trijyā for the reason that it is the H. sine of 3 Rasis or 90°. A Rasi is equal to 30° because the ecliptic circle of 360° is divided into 12 Rasis Mesha, Vrishabha etc. meaning

Aries, Taurus etc. The names of the Rasis in Sanskrit and the modern English words we use for them have the same meaning, which raised a suspicion in the minds of many orientalists that the Hindu Astronomy drew upon the Greek. Many scholars of India assert that the Greeks derived this knowledge from the ancient Hindus; but we shall not enter into the controversy here. It may be noted also that the Sanskrit names of week-days have the same meaning as Sunday, Monday etc.

On the basis of taking trijyā equal to 3438′, the other H. sines or half-chords are also expressed in minutes. Generally twenty-four H. Sines are given in a quadrant and to obtain the H. sine of an angle intermediate, a formula for interpolation also is given. Also the method of calculating the H. sines for every degree is given, as we shall see shortly. In the table of 24 H. sines, the first is H. sine 3°–45′ or H. sine 225′ and this is approximately taken as 225′ because in fig. 6 if $A\hat{O}B = 3°–45'$, the H. sine BC will be almost equal to the arc AB. The H. Vers-sines are also given to get the corresponding H. Cosines easily, for, H. Vers sine 3°–45 = R — H. Cos 3°–45 = 7′ means H. Cosine 3°-45′=3431=H. sine (90°-$3\frac{3}{4}$°)=H_{23} where we use the notation H_r to mean the rth H. Sine.

Now we propose to give here some essential formulae used by the Hindu astronomers, as given in the golādhyāya by Bhāskara under the caption jyotpatti-krama. Incidentally it may be noted that H. sine θ = R sine θ where sine θ is the modern sine of the angle θ°. Similarly H. Cos θ=R Cos θ and H vers θ = R vers θ. Thus when we have an equation of the type. $\sin \delta = \sin \phi \cos z + \cos \phi \sin z \sin a$ in modern astronomy arising out of the famous spherical triangle PZS where P is the Celestial Pole, Z the Zenith and S the position of the Sun or a Star, the Corresponding Hindu formula would be $R^2 H \sin \delta = RH \sin \phi$ $H \cos Z + H \cos \phi\ H \sin Z\ H \sin a$. Occasionally the

radius is taken to be 120, and the corresponding H sines are called Laghu-jyās or simpler H sines used where great accuracy is not required. Srīpati took the radius to be 3270 units in addition to 120 as did Brahmagupta. Munjāla took 488′ and some others some other values also. Out of these 3438′ alone has a right significance (Vaṭēswara took 3272)

Bhāskara says under verses 1 to 5 under Jyotpatti-Vāsanā in the Golādhyāya that the Hindu astronomers got the values of the main H sines of 30°, 45°, 60°, 18° and 36° by inscribing regular polygons in a circle. They are called the pancha-jyakās or the fundamental H sines. From these the others were calculated according to the methods given by Bhaskara as follows.

To start with, we have the fundamental formula

$$H\sin^2\theta + H\cos^2\theta = R^2 \text{ (from fig-6)} \qquad \text{I}$$

In addition to this formula, Bhāskara gives another formula (verse 10, 11 Ibid) H Sin $\theta/2$

$$\overline{\theta + H\text{ vers}^2\,\theta} =$$

In the commentary under the above verses, he has given the method by which II was obtained (Ref. fig. 7) BM = H sin θ where $\widehat{AOB} = \theta$; also AM = H vers θ and $AB^2 = AM^2 + MB^2$. Let N be the mid-point of AB.

$$= AN = H\sin\theta/2$$

$$H\sin\theta/2 = \tfrac{1}{2}AB = \tfrac{1}{2}\sqrt{AM^2 + M}$$

$$= \tfrac{1}{2}\sqrt{H\sin^2\theta + H\text{ vers}^2\,\theta}$$ which

proves the first part of II. Again from the right—angled triangle ABC, $AB^2 = AM\cdot AC = H\text{ vers}\,\theta \times 2R$

$$= \tfrac{1}{2}\sqrt{2R\,H\text{ vers}\,\theta} = \sqrt{\,}$$

which proves the second part.

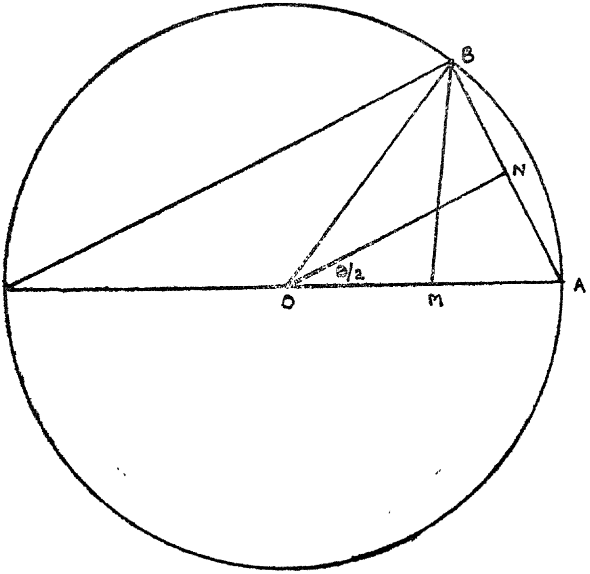


Fig. 7

In the Commentary under verses 1—25 ibid, Bhāskara tells us how formulae I and II are used to construct the table of 24 H sines. To start with, the four H sines of 30°, 45°, 60° and 90° which may be denoted by the symbol Hr where $r=8$, 12, 16 and 24, are known. Now using the formula II, H_4 is obtained from H_8, H_2 from H_4 and H from H_2. Similarly from H_{12}, H_6 and H_3 are successively obtained. Now using formula I, H_{20}, H_{22}. H_{23}, H_{18}. H_{21} are obtained respectively from H_4, H_2, H_1, H_6 and H_3. Now again from H_{20}, H_{22}, H_{18}, we obtain using formula II H_{10}, and H_5, H_{11}, H_9 respectively. Formula I gives again H_{14}, H_{19}, H_{13}, H_{15} from the above. H_{14} gives H_7 and H_7 gives H_{17} using formula II and I respectively. Thus the table is Completed.

Then Bhāskara poses the problem as to how a table of the H sines could be computed when a quadrant is divided into 30 equal parts. He says that formula I and II do not suffice in this behalf and shows how they do not, as follows

in the same commentary cited above. To start with, the H sines of 18°, 30°, 36°, 45°, 54°, 60° are known. They are respectively H_6, H_{10}, H_{12}, H_{15}, H_{18}, H_{20}. Also H_{30} i.e. H sin 90° = R is also known. Formulae I and II will help us to derive,

from H_6, H_3 and from H_3, H_{27} ; also from H_6, we derive H

From H_{10}, H_5 and from H_5, H_{25} and again

from H_{18}, H_9 and from H_9, H_{21} are derived.

The remaining H sines sixteen in number cannot be got from either of the formulae.

To meet the situation Bhāskara gives other formulae of his own discovery as he says "प्रवक्ष्येऽथ विशिष्टमस्मात्" i.e. "I shall tell something more than this. These formulae he gives in the verses 12 to 15. They are

$$\text{Sin}\left(\frac{90 \pm x}{}\right) = \sqrt{R^2 + R\ H\ \text{Sin}\ x}$$

$$\overline{\sin x + H \sin y)^2 + (H \cos x - H \cos y)^2} \qquad \text{IV}$$

$$\sqrt{\frac{(H\ \text{Cos}\ x - H\ \text{Sin}\ x)^2}{2}} = H\ \text{Sin}\ (45 - x) \qquad \text{V}$$

$$R - \frac{2\ H\ \text{Sin}^2 x}{R} = H\ \text{Sin}\ (90 - 2x) \qquad \text{VI}$$

These formulae correspond to the modern formulae

$$\text{Sin}\left(\frac{x - y}{2}\right) = \sqrt{\frac{(\text{Sin}\ x + \text{Sin}\ y)^2 + (\text{Cos}\ x - \text{Cos}\ y)^2}{2}}$$

$$\frac{(\text{Cos}\ x - \text{Sin}\ x)^2}{2} = \text{Sin}\ (45 - x)$$

$$1 - 2\ \text{Sin}^2 x = \text{Cos}\ 2x \qquad \text{respectively}$$

These formulae imply a knowledge of the expansion of Sin $(x \pm y)$ which is given in verses 21, 22 in the form

$$H \text{ Sin } (x \pm y) = \frac{H \text{ Sin } x \; H \text{ Cos } y \pm H \text{ Cos } x \; H \text{ Sin } y}{R}$$

The formula H Cos $(x \pm y)$ is got from VII by putting $90 - (x \pm y)$ for $x \pm y$.

To construct the remaining sixteen H Sines Bhaskara directs us to use his formula IV wherein taking $x = 27°$, and $y = 15°$, we have H_2 which gives H_{28}. From H_{28} we have H_{14}, H_7 and H_1 from H_2. Then H_{16}, H_{23} and H_{29} are got from H_{14}, H_7, and H_1 respectively. From H_{16} again we have H_8, and H_4 which in turn give give H_{22} and H_{26}. H_{26} gives H_{13} which in turn gives H_{17}. H_{22} similarly gives H_{11} which in turn gives H_{19}. Thus the table is complete.

Verses 16 *to* 20. (*Ibid.*) give us the method of constructing the table of 90 H sines in a quadrant, through the formula H Sin $(x \pm 1)° =$

This formula is got evidently by interpolating from his knowledge of H Sin 3° and H Sin $3\frac{3}{4}°$. He also gives that H Sin $5°\frac{3}{4}$ is more correctly equal to 224′, 51″. In the table of 24 H Sines, he gives the formula $H_{r+1} = Hr\left(1 - \frac{1}{497}\right) + H \text{ Cos } x_r \times \frac{100}{1529}$ which could be similarly got by interpolation.

The determination of H Sin (A+B) from H Sin A, H Sin B, H Cos A, H Cos B is called Samāsa-Bhāvanā and that of H Sin (A−B) is called Āntara-Bhāvanā, whereas computation of H Sin 2A from H Sin A and H Cos A is called Tulyabhāvanā. The word Vajrābhyāsa is used for 'Cross-multiplication' in this context.

We shall now prove how the formula for H Sin $(x+1)°$ led Bhāskara to arrive at the differential formula

δ (Sin x) = Cos x δx. Since H Sin $(x+1)^\circ$ = H Sin x + $\dfrac{\text{H Cos x} \times \text{H Sin } 1^\circ}{R}$ approximately, H Sin $(x+1)^\circ$ − H Sin x° = H Cos x $\times \dfrac{\text{H Sin } 1^\circ}{R}$ = H Cos x × a constant. Hence Bhāskara could see that the variation in the function H Sin x is proportional to H Cos x. Let it be now required to find the increment in H Sin x for an increment δx in x where $\delta x < 60'$. Let H Sin $(x+1)^\circ$ − H Sin x = $\dfrac{60' \times \text{H Cos x}}{R}$ = y where y is called the Bhogya-Khanda. Then Bhāskara argues "If for an increment of 60′, there is an increment of y, what shall we have for δx ?". The answer is $\dfrac{y\delta x}{60} = \dfrac{60 \times \text{H Cos x}}{R} \times \dfrac{\delta x}{60} = \dfrac{\text{H Cos x } \delta x}{R}$.

Hence H Sin $(x + \delta x)$ − H Sin x = δ (H Sin x) = H Cos x $\times \dfrac{\delta x}{R}$ which corresponds to δ (Sin x) = Cos x δx. Bhāskara is thus the first mathematician to have perceived this differential formula 500 years before Newton and Leibnitz.

In the context of the preparation of the table of 24 H Sines which was there in Āryabhatiya as well as Sūryasiddhānta, we have to offer the following remarks. The method of the construction of this table was the subject-matter of some study by S. N. Naraharayya and A. A. Krishnaswami Ayyangar[1]. In this study it was supposed that the method was based on finite differences according to the verses of the Sūryasiddhānta 15 and 16 of chapter II. The articles referred to reveal the difficulty in constructing the sine-table following this method. Some subsidiary corrections in the verse "एकविंशाश्च विंशाश्च

1. S. N. Naraharayya—Journal of Indian Math. Society, Vol. XI—First Series Pages 105–113.

2. A. A. Krishnaswami Ayyangar—J. I. M. S., Vol. XV—First Series Pages 121–126.

quoted from the Brahma Siddhānta by Ranganātha in his commentary of Sūryasiddhānta were alluded to in the articles cited but no satisfactory mathematical explanations were given by them. We shall give hereunder a satisfactory explanation of the matter discussed in the articles.

In the first place it may be noted that in the table of those 24 H sines, the sixteenth as given by Bhāskara namely 2977 is more correct than that given in the Sūryasiddhānta namely 2978 (Lakshmi Venkateswara press edition 1955 Bombay).

In the course of the Commentary under the verses 15, 16 of the Sūryasiddhānta, Ranganātha gives the hint which must have been at the back of the mind of the author of the Sūryasidhhānta when he gave the rule to construct the table cited. Just as $\delta\,(\text{H Sin}\,\theta) = \frac{\text{H Cos}\,\theta\delta\theta}{\text{R}}$, Similarly the formula $\delta\,(\text{H Cos}\,\theta) = \frac{-\,\text{H Sin}\,\theta\,\delta\,\theta}{\text{R}}$ must have been known to the author. The negative sign means that the successive differences of the H sines namely 225, 224, 222, 219 etc. are decreasing and also that the successive differences of these differences are increasing according to the H sine. Just as Bhāskara could see that the H sines were increasing and the successive differences of the H sines were in Kotijyānupāta i.e. in direct ratio to the H Cosine at their respective place, similarly, the author of the Sūryasiddhānta could see that the second differences cited above were in Kramajyānupāta as hinted by Ranganātha.

From the formula $\delta\left(\frac{\text{H cos}\,\theta\,\delta\,\theta}{\text{R}}\right) = \frac{-\,\text{H sin}\,\theta\,\delta\,\theta^2}{\text{R}^2}$ putting $\theta = 90°$, we have the second difference numerically

$$R \qquad \times \qquad = 14' - 43'' - 30''$$

Ranganātha made a mistake in taking this to be

$= 15' - 16'' - 48'''$ – Even $\frac{\delta\theta^2}{R}$ is approximate and a more correct value of the second difference would be 14' — 47" approximately. Ranganātha then argues that taking this second difference to be 15 for the H sine 3438 ' what will it be for the H sine 225′ ? ' The answer would be $\frac{15 \times 225}{3438} = \frac{15 \times 25}{382} = \frac{375}{382} = 1'$

So, the second difference in the beginning of the table happens to be $1'$ ie $\frac{225}{225}$. This led the author of the Surya siddhānta to use the words "तद्विभक्तलब्धोन". This being an approximate formulation, naturally necessitated a second formulation where the approximation led to an error of 1' through the verse "एकविंशाच्च विंशाच्च etc." This second formulation intended to make a correction, was done in the wake of a correct calculation through the formulae I & II which were known even prior to Bhaskara.

Verses 10, 11. To find the H sine of an intermediate angle. Suppose it is required to find the H sine of an angle $\theta°$ ie $\theta \times 60'$. Divide this by 225; the quotient gives the previous H sine. Then $\frac{R \times D}{225}$ where R is the remainder, and D the difference between the previous and next H sines, added to the previous H sine gives the H sine required.

Comm. The formula is evidently based on an application of rule of three.

Verse 11. To find the angle when the H sine is given Suppose the H sine of an angle is given to be x′. Subtract the greatest H sine that could be subtracted from this.

Suppose the H sine of $\theta°$ could be subtracted. Let the remainder be r. Then $\frac{r \times 225}{D}$ where D is the difference between the previous and next H sines, added to θ gives the angle corresponding to x′.

Comm. Evidently this is the converse of the previous process and this also is based on Rule of three.'

Verses 12—15. The H sine of the obliquity of the ecliptic taken to be 24° is 1397. Now, the successive differences of the H sines will be given (on the basis of taking R = 120) which are known as Laghu-Jyās intended for ease in Computations, namely 21, 20, 19, 17, 15, 12, 9, 5, 2. These are given for intervals of 10°, so that if it be required to find the H sine of x°, let q be the quotient and r the remainder when x is divided by 10. q gvies the number of the previous H sine. Then $\frac{r \times D}{10}$ where D is the next difference or jyākhanda as it is called, added to the previous H sine gives the required H sine. In this table the H sine of 24° is 48′ — 45″. Also the H versines in this table are got by the reverse differences. To get the angle $\theta°$ for a given H sine say x′ subtract the sum of as many differences (Jyā—Khandas) as could be from x. Let the remainder be r. Then $\frac{r \times 10}{D}$ where D is the next jyā—Khanda added to the previous angle upto which the jyākhandas have been subtracted, gives the required angle. The H sine will be more accurate if the Bhōgya-Khanda or the next H sine—difference is rectified (as per the rule of interpolation next given).

Comm. H vers θ = R—H Cos θ = R—H sin (90—θ) so that H verse $3\frac{3}{4}°$ = 3438—H sin ($86\frac{1}{4}°$) = 3438—3431 = 7 as given in the previons table. Similarly in the above table of Laghu-Jyās, H vers 10° = R — H cos 10° = 120 — H sin 80° = 120 — (21 + 20 + 19 + ... + 5) = 2 so that the

above differences in the reverse order give the H versines. The rest of the contents of the verses is simple, the processes being based on the 'Rule of three'.

Verse 16. Rectification of the next H sine difference known as Bhōgya—Khanda.

The difference of the previous and the following H sine—differences being multiplied by the remaining degrees and divided by 20, the result is subtracted from the arithmetic mean of the previous and following H sine-differences to give the rectified H sine—difference, in question.

Comm. This is a formula for interpolation which agrees with the interpolation formula given by Ball in his spherical astronomy on page 18 in the form $y = yo + \frac{x}{h} (y_1 - yo) + \frac{x(x-h)}{2h^2} (y_2 - 2y_1 + yo)$. This formula is a re-statement of the formula enunciated by Brahmagupta in his work Brahma Sphuta Siddhānta as well as Uttara-Khandakhādya in the form "गतभोग्यखण्डकान्तरदलविकलवधात् शतैर्नवभिराप्तैः, तद्युतिदलं युतोनं भोग्यादूनाधिकं भोग्यम्" wherein in the place of ten-degree-interval, a fifteen-degree-interval ie 900′-interval was taken. Rule of three is a linear formula of interpolation, whereas the above is a quadratic formula reflecting much credit on the mathematical genius of Brahmagupta.

We shall now see how the formula is applied and what mathematical significance it has. Suppose it is required to find the H sine of 24° from the previous table of H sine—differences given for intervals of 10° — from the table H sin 10° = 21, H sine 20 = 41, H sine 30 = 60 where R is taken to be 120. Now to find H sin 24°, we are asked to rectify the next H sine—difference namely 19′, where the table is 21, 20, 19 etc. As a first approximation, applying rule of three H sin 24° $= 41 + \frac{4}{10} \times 19 = 48{\cdot}6 = 48' - 36''$.

This is a crude approximation, the actual value being 48′ — 48″ — 14‴. Application of rule of three is justified if the H sine—differences are uniform, but they are not so, being in a decreasing order. So, the following H sine—difference namely 19' is to be rectified so as to be applicable at 24°. In other words we have to take such a H sine—difference which will hold good at 24°, not at 20° or 30°—from 10° to 20°, the H. S. d. (H Sine —difference) is 20', and from 20° to 30° it is 19'. If that be so what will be exactly at 24°? If should be less than 20' and greater than 19'. Now the argument advanced by Bhāskara is that the H Sine —difference at the mid-point of the 2nd and 3rd differences namely 20 and 19 should be $\frac{20 + 19}{2} = 19{\cdot}5$.

The H sine —difference at the end of the third interval is 19. Then by the rule of three 'If there is a decrease of $19{\cdot}5 - 19 = {\cdot}5$, during the course of the 10° of the third interval, what should the difference be for 4°? (where we have to find the H sine at 24°) The result is $\frac{4}{10} \times {\cdot}5 = \frac{4 \times 1}{20}$ which is given by the words "यातैष्ययोः खण्डकयोर्विशेषः शेषांशनिघ्नो नखहृत्". This decrease makes the H sine—difference at 24°, $19{\cdot}5 - \frac{4}{20} = 19{\cdot}3$ at 24°—Now the argument to find the H sine at 24° is 'If for an interval of 10°, the H sine—difference is 19·3, what should it be for 4°?' The answer is $\frac{4}{10} \times 19{\cdot}3 = 7{\cdot}72$. Hence the H sine of 24° is $21 + 20 + 7{\cdot}72 = 48{\cdot}72 = 48' - 43'' - 12'''$ which is nearer the truth 48′ - 48″ - 14‴ than what was obtained by the crude rule of three namely 48′ - 36″.

Here we have to explain Bhaskara's words more elaborately, because, Kamalākara happened to criticise Bhaskara's words in this context. Bhaskara Says "The H sine difference at the end of an interval is the arithmetic mean of the preceding and succeeding differences, whereas the succeeding one is that which holds good at end of the succeeding. In between, we have to apply the rule of three to

obtain the rectified difference." What Bhāskara means is this. At the end of an interval, to obtain the H sine, it is enough to add the H sine difference belonging to that interval to the preceding differences. But when it is required to find the H sine in the interior of an interval, we have to construe that the difference at the end of the previous interval is the arithmetic mean of the previous and succeeding differences. There is apparently a self-contradiction in Bhāskara's words; for, at the end of the interval, according to his own words, the difference is that belonging to the previous interval and not the arithmetic mean as postulated. The contradiction will not be there when we read Bhaskara's mind that he means "When we require to find the H sine in the interior of an interval only, the difference at the beginning of that interval is to be taken as the arithmetic mean of the previous and the current differences, and that at the end of the interval the current difference holds good."

The truth of Bhāskara's statement could be seen analytically as follows. The arithmetic mean of the preceding and succeeding H sine differences is (The context is to rectify the third H sine difference namely 19, for, we were finding the H sine of 21°) $\frac{AB + BC}{2}$ (Ref fig. 8) where OA, AB, BC etc are the successive differences.

$$\frac{AB + BC}{} = \frac{H \sin 20 - H \sin 10 + H \sin 30 - H S 20}{}$$

$$= \frac{H \sin 30 - H S 10}{2} = R\left(\frac{\sin 30 - S 10}{2}\right)$$

$$= R \times \cos 20 \sin 10 = \frac{H \cos 20 \, H \sin 10}{R}$$

The numerical difference of the preceding and succeeding H sine differences is (i.e. यातैष्यखण्डविशेष). AB − BC = (H sin 20 − H S 10) — (H sin 30 — H sin 20) =

$$\frac{2\,H\,Cos\,15\,H\,sin\,5}{R} - \frac{2\,H\,Cos\,25\,H\,sin\,5}{R} = \frac{2\,H\,sin\,5}{R}$$

$$(H\,Cos\,15 - H\,Cos\,25) = \frac{2\,H\,sin\,5}{R} \times \frac{2\,H\,sin\,20\,H\,sin\,5}{R}$$

Let now X be the point where we are to find the H sine (Here let us take it as x° after the previous interval for generalisation). Then Bhaskara's formula would give

$$\frac{H\,Cos\,20\,H\,Sin\,10}{R} - \frac{2\,H\,sin\,5}{R^2} \times \frac{2\,H\,sin\,20\,H\,sin\,5}{20} \times x$$

$$= \frac{H\,Cos\,20\,H\,sin\,10}{R} - \frac{x}{10\,R^2}\,2\,H\,sin\,20\,H\,sin^2\,5$$

$$= \frac{2\,H\,Cos\,20\,H\,sin\,5\,H\,Cos\,5}{R^2} - \frac{1}{R^2} \times \frac{x}{10} \times 2\,H\,sin\,20\,H\,sin^2\,5$$

$$= \frac{2\,H\,sin\,5}{R}\left\{\frac{H\,Cos\,20\,H\,Cos\,5 - \frac{x}{10} \times H\,sin\,20\,H\,sin\,5}{R}\right\}$$

put now successively x = 0° and 10° to get the rectified differences at B and C respectively; then those rectified differences would be respectively H Cos 20 H sin 10 and $\frac{2\,H\,sin\,5}{R} \times H\,Cos\,25$. But $\frac{AB+BC}{2} = \frac{H\,Cos\,20\,H\,sin\,10}{R}$ (found above) and BC = H sin 30 − H sin 20 = $\frac{2\,H\,Cos\,25\,H\,sin\,5}{R}$ In other words the rectified differences at B and C are respectively what exactly has been stated by Bhāskara. Hence Kamalākara's condemnation of Bhāskara is quite unjustified.

Verse 17. To rectify the arcual difference to obtain the arc for a given H sine.

Subtract as many H sine-differences as could be subtracted from the given H-sine. Half of the remainder multiplied by the difference of the preceding and succeeding H sine-differences and divided by the succeeding and

the result being subtracted from or added to as the case may be (added in the case of Hversines) the arithmetic mean of the preceding and succeeding H sine-differences gives the rectified H sine-difference while finding the arc for a given H sine.

Comm. Let the given H sine be that of 24° found before i.e. 48·72. We could subtract 21 and 20 from this and the remainder is 7·72; half of this is 3·86 which multiplied by (20–19) is 3·86. This divided by the succeeding difference namely 19 is ·2 approximately. The arithmetic mean of the preceding and succeeding si $\frac{20+19}{2}$ $= 19{\cdot}5$. If the above result is subtracted from this, we have $19{\cdot}5 - {\cdot}2 = 19{\cdot}3$. If for 19·3 we have 10° increment what shall we have for 7·72. The answer is $\frac{10 \times 7{\cdot}72}{19{\cdot}3}$ $= 10 \times {\cdot}4 = 4°$. Hence the required arc is $20° + 4° = 24°$.

The proof is analogous to the previous proof. Having subtracted 21 and 20, the remainder is 7·72, (Ref. fig. 8)

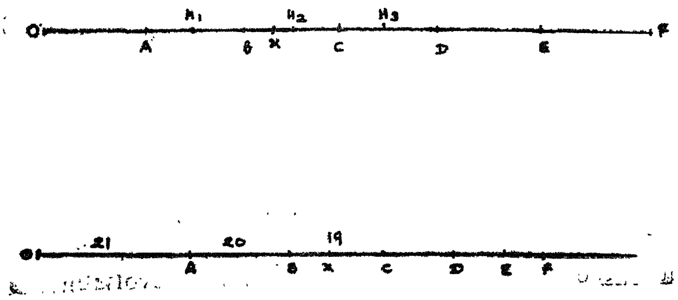


Fig. 8

so that BX = 7·72. Now during the course of the succeeding interval of 19, there has been a decrease of $19{\cdot}5 - 19 = {\cdot}5$ or to put it in general terms, during the course of the succeeding interval BC there has been a decrease

$\frac{x+y}{2} - y = \frac{x-y}{2}$ where x is the previous interval AB $\frac{x+y}{2}$ is the H sine difference at B. Hence the argument is "If for y, there has been a decrease of $\frac{x-y}{2}$, what will it be for δ (Here $\delta = 7{\cdot}72$)"? The answer is

$\delta \frac{(x-y)}{2} \times \frac{1}{y} = \frac{\delta}{2}(x-y) \times \frac{1}{y}$ = अद्दशेपकार्धम् × गते

This result is to be subtracted from 2

i.e. ·2 is to be subtracted from 19·5 to give the rectified H sine–difference.

Verse 18. Definition of Kēndra and assignment of sign thereto.

The excess of the longitude of the mean place over that of the apogee or aphelion as the case may be is called the mean anomaly. The excess of the longitude of the point called S'īghrōccha over that of the planet rectified by the first equation known as Manda-phala or equation of centre is known as the Sīghra-anomaly. The equation of centre is positive or negative according as $180 < m < 360$ or $0 < m < 180$ where m is the mean anomaly. The case will be reverse in the case of the sign of Sīghraphala, the second equation.

Comm. "चन्द्रसूर्यौ स्फुटौ स्यातां मान्देनैकेन कर्मणा" i.e. The Moon and the Sun could be rectified by the equation of centre alone. This means that the Moon revolving round the Earth directly and that the Sun revolving relatively round the Earth are subject to only one correction namely the equation of centre for rectification. In the case of the Sun, though the fact is that he is revolving round the Earth relatively, assuming as Hindu astronomy does that he is revolving directly round the Earth and him to the correction of the equation of centre

does not alter the mathematics that goes into his rectification. According to modern astronomy the Sun and the Moon, one relatively and the other directly go round the Earth, in ellipses, the Earth being in one focus whereas in Hindu astronomy both the Sun and the Moon are taken to be going round the Earth in eccentric circles i.e. circles whose centres do not coincide with the centre of the Earth. In fact, Bhāskara says in so many words "भुमेर्मध्यं खलु भवलयस्याऽपि मध्यं यतः स्यात्, यस्मिन् वृते भ्रमति खचरो नाऽस्य मध्यं कुमध्ये, भूस्थो द्रष्टा न हि भवलये मध्यतुल्यं प्रपश्येत्, तस्मात् तद्ज्ञैः क्रियत इहतत् दोःफलंमध्यखेटे i.e. The centre of the celestial sphere coincides with the centre of the Earth. The centre of the circle in which a planet goes does not coincide with the centre of the Earth. Hence an observer on the surface of the Earth finds the True planet's position differing from that of the mean planet, so that what is called the correction of Bhujaphala is to be made in the mean position of the planet to get the True position." Here it has to be noted that the Bhujaphala mentioned stands both for the equation of centre and the second equation known as S'īghraphala as well. One may wonder how it could be so, but it may be noted that in the formulation of both the equations, the centre of the eccentric does not coincide with that of the Earth and also in both the cases the equation contains the term H sine of the anomaly where the word anomaly whether it be of the first or second equation is known as 'Bhuja'.

In the case of the other planets, the fact is that they go round the Sun in elliptic orbits, the Sun being in one focus. In Hindu Astronomy, we shall see that the centre of the eccentric circle in which these planets are taken to revolve coincides with the Sun. Hence, though the ancient Hindu Astronomy postulates geocentric motion, the mathematics that goes into the formulation of the second equation, makes the Sun's centre the centre of planetary revolution. One may wonder again how the equation of centre

formulated by Hindu Astronomy agrees with its formulation in Modern Astronomy which enunciates elliptic motion; but it will be seen that the eccentric-circle theory also gives very approximately the same formula for the Equation of centre. There is just one point of difference, which does not matter. Whereas in Modern Astronomy the mean anomaly is reckoned from the perigee or perihelion as the case may be, it is reckoned in Hindu Astronomy from the apogee or aphelion. The difference is made up by prefixing the appropriate sign to the equation. The word Mandōccha stands for the apogee in the case of the Sun and the Moon and for the aphelion in the case of the other planets and the word Manda kendra stands for the mean anomaly in both the cases. In Hindu Astronomy the word 'graha' stands for not only the five planets Mercury to Saturn but also for the Sun and the Moon. Why that word is applied to the Sun and the Moon as well is, that both the Sun and the Moon also while moving among the stars along with the five other planets Mercury to Saturn, wield an influence on the residents of the Earth. The etymology of the word 'graha' is गृह्णातीति वा गृह्यते अनेनेति वा ग्रहः ie. 'that which seizes upon the fates of the residents of the Earth', with this etymological significance only, even the lunar orbital nodes known as Rāhu and Kētu are also taken to be grahas in Hindu Astronomy. Hence translating the word graha as a planet and criticising Hindu Astronomy for taking the Sun, Moon and the lunar orbital nodes also as such is not right. In other words the translation should be pronounced wrong. Uranus, Neptune and pluto were not mentioned in Hindu Astronomy.

We shall now elucidate the eccentric and epicyclic theories of Hindu Astronomy, which will be seen to give identical position to the planets. How they came to be postulated will be also elucidated. Incidentally we deal with the 'Bhagaṇōpapatti' or the proof of the numbers of sideral revolutions of the various grahas enunciated in the

beginning of the Madhyādhikāra, Bhagaṇādhyaya in verses 1 to 6. Even Bhāskara gave such a proof as appealed to Āgama ie. 'Authority', which proof therefore will not be acceptable to a student of Modern Astronomy, who is likely to question how the Āgama came into existence.

How the Āgama came to formulate the number of sidereal revolutions of the grahas, we shall now see.

The forefathers of Hindu Astronomy (have been reported to be eighteen in number in the famous verses "सूर्यः पितामहो व्यासो वसिष्ठोऽत्रिः पराशरः कश्यपो नारदो गर्गो मरीचिर्मनुरङ्गिराः, लोमशः पौलिशश्चैव च्यवनो यवनो भृगुः, शौनकोऽष्टादशैतेस्युः ज्योतिश्शास्त्रप्रवर्तकाः" Of these Brahmagupta mentions Brahma-Siddhānta, which he reports to have resuscitated. Varāha Mihira gave a version of the old Sūryasiddhānta, mentioning that it accorded with observations. Āryabhata says that he revived his system from the then existing ocean of knowledge both good and spurious. The fact that none of these outstanding astronomers mentioned that they had derived their systems from a foreign source, and the reasonableness in presuming that all these three could not be impostors, make the author of this work feel strongly that there should have been some works in the name of Āgamas extant long before these Acharyās. The argument that the crude Vedānga-jyotiṣa alone should have existed before Āryabhata, simply because, no other work worth the name has been discovered, may not be correct. It is quite possible that crude works could exist side by side with advanced scientific works, just as even nowadays we have thinkers and works of a primitive type existing along with highly advanced thinkers as well as scientific works).

Bhagaṇopapatti. In the first place, the forefathers of Hindu Astronomy must have noticed very easily that the Moon has a motion among stars, for, this could be detected

even by a lay man during the course of a single night. So, the period of a single sidereal revolution, could be roughly recognized by noticing the conjunction of the Moon with a luminous star. Having thus observed a good number of sidereal revolutions, which could be done even with the naked eye, the average period could be arrived at with sufficient accuracy within the course of a few years. Having thus obtained almost accurately the average of a sidereal revolution of the Moon, the sidereal revolution of the Sun could have been arrived at as follows. The moment of an eclipse solar or lunar could be observed with the naked eye. Observing a good number of eclipses within the course of a few years, the average of a lunation could be easily arrived at very accurately, for, in between the eclipses of the same nature an integral number of lunations elapse. That the Sun also has a motion among stars must have been noticed clearly during the course of a few months, for, observing at Sunset the star that was rising, it should have been noticed gradually even during the course of a month, that the Sun must have been approaching the star or vice versa and as the stars were found to keep the distances amongst them constant, it was the Sun that was approaching the star and not the star it was that was approaching the Sun. Having decided thus that the Sun was moving among stars from west to East, the approximate period that the Sun took to complete a sidereal revolution was arrived at. Then, as both the Sun and the Moon were having east ward motion, and as the Moon has a more rapid motion, the arc by which the Moon overtakes the Sun during a day was roughly noticed. Thus arriving at a rough estimate of a lunation within which a conjunction of the Moon with the Sun recurs, it was noticed that the excess of the sidereal revolutions of the Moon over those of the Sun gave the number of lunations. Since a correct estimate of both a sidereal revolution of the Moon as well as that of a lunation were previously arrived at, the number of sidereal revolutions of the Sun during

the course of a certain period were computed whereby a correct estimate of a sidereal solar year was arrived at. This period could also be checked simultaneously by observing the interval between the heliacal risings or settings of a particular star of the Zodiac as well. Thus far, we have seen how the sidereal periods of the Sun and the Moon were determined very accurately. It may be noted that these periods as determined by the Hindu astronomers were correct to a good number of decimal places.

When once the Sun and the Moon were found to be having eastward motion among stars, and when it was discovered that there were other luminous bodies like the Jupiter and Venus etc. moving among stars, it was attempted to determine their sidereal periods. It must have been done as follows. In the first place, it was noticed that these other luminous bodies which were five in number, namely Mars, Mercury, Jupiter, Venus and Saturn, were found to be having retrograde motion also unlike the Sun and the Moon. As these five bodies were looking like stars they were named Tārā-grahas i.e. grahas looking like stars. Also a distrinction could be drawn very easily between Mercury and Venus on the one hand and the other three on the other, for, the former were always found oscillating about the Sun, never parting from him through long distances. Thus during the course of a sufficiently long interval, the geocentric sidereal periods of Mercury and Venus coincided with that of the Sun. In other words, it was taken that the geocentric sidereal periods of Mercury and Venus also were taken to be an year. This is clear from the statements made by all the Siddhāntas that in a Kalpa of 432000000 years the Sun, the Mercury and Venus all the three make 432000000 sidereal revolutions. Then with respect to the other three planets Mars, Jupiter and Saturn, it was noticed that they were having a pre-dominantly longer period of direct motion, though there was a retrograde motion for some time. This

gave the clue to arrive at an approximate estimate of their sidereal revolutions. But the correct estimates were arrived at not by observing their conjunctions with stars, for, that would take a very long period of observation in the case of Saturn, but, by observing a good number of their heliacal risings or settings. The interval between two consecutive heliacal risings or settings being a little greater than an year, ten or fifteen observations could be done very easily by a single person. Then the aforesaid argument given in the case of finding the sidereal revolution of the Sun, was also advanced in the case of these three planets Mars, Jupiter and Saturn. Let x° be the arc that the planet covers during the course of a day. Let a° be the arc covered by the Sun during the same period which was previously known and that correctly. So, during the course of a day the Sun overtakes the planet by $(a - x)^\circ$. Hence to overtake 360°, the period S was computed. This period was observed as the interval between two consecutive heliacal risings or settings and known as the synodic period. So, from the equation $\frac{360}{a - x} = S$, the value of x could be arrived at, wherefrom p the sidereal period was determined. This sidereal period could be also determined in another way. Noting the distance covered among stars by a planet during the course of a synodic period, using rule of three, the sidereal period could also be arrived at with a good accuracy, for, the retrograde motion affects equally each synodic period. An average of such determinations made in two ways could give the sidereal periods of Mars, Jupiter and Saturn with a good amount of accuracy. It will be noted here, that the average of a good number of geocentric sidereal periods in the case of these planets (called Superior) is also the heliocentric sidereal period (for a proof of this statement reference may be made to page 80 of the author's '*A critical etudy of the ancient Hindu astronomy*, published by the Karnatak University Dharwar). It is why the sidereal periods of these planets

as given by Hindu astronomy tally with the heliocentric sidereal periods given by modern astronomy. This is also one of the reasons why heliocentric motion of the planets could not be detected by Hindu astronomy, and also why a statement was made that "In the case of Mercury and Venus the Sun was the planet. and they are termed as S'ighrōcchas, whereas in the case of Mars, Jupiter and Saturn, they are the planets while the Sun plays the part of S'īghrōccha "कुजजीवशनीनां तु रविः शीघ्रोच्चनामकः, ज्ञशुक्रयोः तौ शीघ्रनामकौ" (An elucidation of this statement will be given shortly)

In the case of the planets Mercury and Venus (Inferior planets) one may wonder how under the geocentric theory, their heliocentric periods could be arrived at, though they were not recognized as such but were pronounced as the "geocentric periods (not considered as heliocentric)" of two points known as their S'ighrōcchas. Here we come across the peculiar concept of a S'ighrōccha which arose out of the fact of postulating a geocentric system in the place of the heliocentric. This concept is to be elaborated, in as much as confusion is there in the minds of many interpreters of Hindu astronomy in this behalf.

In the first place let us consider as to how the rectification of the Sun and the Moon, known as sphutīkaraṇa was achieved. Having got their sidereal periods, their mean daily motions were calculated. Also a period was conceived, during which this Sun and the Moon would perform an integral number of revolutions. This period was termed as a Mahāyuga (or simply a yuga as we hereafter name it) whose duration was estimated as 4320000 solar years. That the yuga is an integral L.C.M. so to say of the sidereal periods of the Sun and the Moon (also of the other planets as we shall see shortly) could be seen from the statement of the Surya Siddhānta युगे सूर्यज्ञशुक्राणां

खचतुष्करदार्णवाः, कुजार्किगुरुशीघ्राणां भगणाः पूर्वयायिनाम्, इन्दो रसाग्नित्रित्रीषुसप्त भूधरमार्गणाः" i,e. 'In a yuga, the Sun the Mercury and Venus perform 432000 sidereal revolutions as well as the S'īghrōcchas of Mars, Jupiter and Saturn, whereas the Moon performs 57753336 revolutions' (It may be recalled here that the Mercury and Venus are oscillating about the mean position of the Sun; also it will be noticed that the Sun playing the part of the S'īghrōcchas in the case of Mars, Jupiter and Saturn, their S'īghrōcchas are also deemed as making the same number of revolutions as the Sun. In as much as the S'īghrōcchas in the case of Mercury and Venus are looked upon as different from the planets, so in the case of Mars, Jupiter and Saturn also the Sīghrocchas are taken as different Divine entities though coinciding with the Sun in position). However smaller periods could be conceived as integral L.C.M's of the sidereal revolutions of the Sun and the Moon, but a presumption sponsored by a sense of orderliness in the Cosmos, that the planets should all have been started from the Zero point of the Zodiac, made the integral L.C.M. to be of such a dimension as 4320000 solar years in which period the other planets also would have made an integral number of sidereal revolutions. Here in this point the traditional Hindu astronomers place their faith in the Āgama, which said that the planets were all started at the Zero-point of the Zodiac in the beginning of the yuga and were ordained to return to the same point at the close of the yuga. Even a rational astronomer like Bhāskara, apparently placing faith in the Āgama, while adducing a proof in the name of Bhagaṇōpapatti, states that after obtaining the mean daily motions of the planets, calculates them for the period of a Kalpa taking it on trust that the planets were started at the initial point of the Zodiac in the begin-

ning of the Kalpa, A modern astronomer, however, questions the assumption that the planets were all started at the first point of the Zodiac, and even though they might all have been in conjunction at that point in some remote past, whether it was the initial point of the reported Kalpa. Proceeding on the basis of the reported initial conjunction of all the planets at the first point of the Zodiac, and calculating the number of days that have elapsed from the beginning of the yuga, the mean positions of the Sun and the Moon were computed. Noticing that these mean positions did not exactly accord with the true observed positions, the ancient astronomers tabulated the differences between those mean and true positions. These differences were found to be zero at two diametrically opposite points, and maximum roughly at two points differing by a quadrant from them. To account for these differences, the thought that occured to their minds was that probably the Sun and Moon did not move in a circle whose centre coincided with that of the Earth but were moving in an eccentric circle i.e. a circle whose centre is at some other point than the Earth's centre. This surmise could be made because unequal motion was accountable only on varying distance from the Earth's centre and a celestial body appearing to move fastest must be nearest whereas the same appearing to move slowest must be farthest. Thus in the first place seeing no reason for non-circular motions and also expecting the celestial bodies to move only in circles, for, a circular motion appealed to them as the most ideal motion, the ancient astronomers later postulated an eccentric circular motion with respect to the Sun and the Moon. This postulation appeared to give good results as seen below.

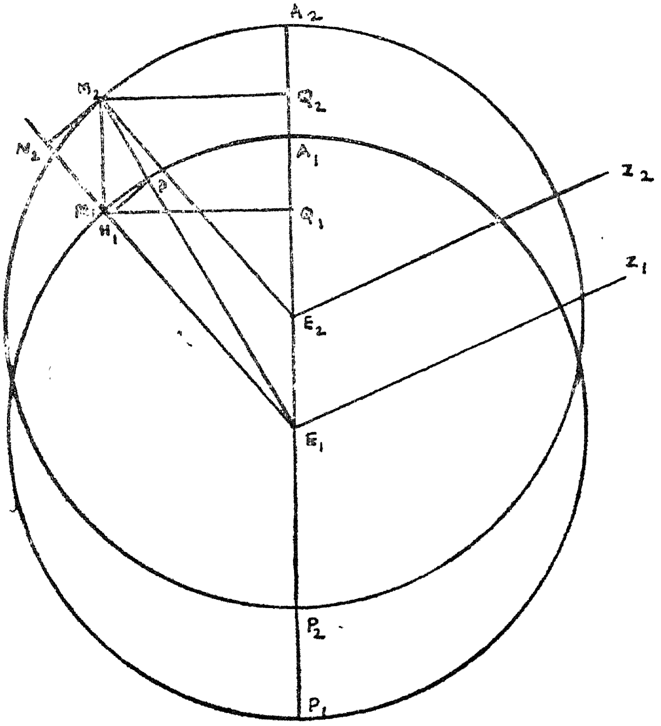


Fig. 9

Eccentric circle theory.—Let E_1 be the earth's centre; let M_1PA_1 be the circular orbit in which the planet (here the Sun or the Moon) moves with a uniform motion. This planet is termed the Madhyagraha or the mean planet. Let $M_2A_2P_2$ be the actual orbit of the planet whose centre E_2 is removed a little away from E_1. Since the centre E_2 is moved in a vertical direction away from E_1, every point of the eccentric circle (E_2) will be vertically over the corresponding point of the mean circle. Thus M_2 will be the position of the actual planet where M_2 is vertically above M_1 the mean planet. Join E_1M_2 to cut the mean circle in P. Since A_2 is the position of the actual planet farthest from E_1 the Earth's centre, the planet should have

the slowest motion there. So this point A_2 is termed Mandoccha, Manda because it is point where the planet is slowest and Uccha because it is the highest or the farthest point from the Earth's centre. Corresponding to this Mandoccha in the eccentric circle A_1 is termed the Uccha in the mean circle. Also p the point where E_1M_2 the line joining the Earth's centre to the actual planet and called the Mandakarna, cuts the mean orbit is taken to be the position where the apparent planet is situated. Thus 'p' is seen to be deflected from the mean planet M_1 towards the Mandoccha on which account the Mandoccha is considered to be attracting the planet '– – गेहा:कर्षको भवति" as Bhāskara puts it. The angle $A_2\hat{E}_2M_2$ is spoken of as the Manda-Kendra or the mean anomaly and it is equal to $Z_2\hat{E}_2M_2 - Z_2\hat{E}_2A_2$ = longitude of the planet minus the longitude of the Mandoccha, where E_1Z_1 and the parallel E_2Z_2 are directions towards the Zero-Point of the Zodiac. This accounts for the statement 'मृदूच्चेन हीनो ग्रहो मन्दकेन्द्रम्' (of the verse under elucidation) i.e. the excess of the longitude of the planet over that of the Mandoccha is termed Mandakendra. While M_1 is termed the Madhya-graha in the mean orbit, M_2 is termed the prativritta–Madhyagraha, and not spaṣṭagraha as might be deemed, while p is spoken of as the spaṣṭagraha or the True planet or apparent position of the planet. The word Prativritta stands for the eccentric circle. Now M_1P the difference between the mean and True positions is spoken of as the Mandaphala which corresponds to the modern 'Equation of Centre'. To find its value draw perpendiculars PN_1 and M_2N_2 on

Triangles $E_2M_2Q_2$ and $M_1M_2N_2$ are evidently similar

Hence $\frac{M_2Q_2}{E_2M_2} = \frac{M_2N_2}{M_1M_2}$ so that $M_2N_2 = \frac{M_1M_2}{E_2M_2} \times M_2Q_2 = \frac{r}{R}$ H sin $M_2\hat{E}_2A_2$ (1) (which is equal to r Sin $\hat{E}_2$ in modern terms) Now in the case of the Equation of centre which is

generally a small quantity M_2N_2 is taken to be equal to PN_1. If, however, this approximation is not made, $PN_1 = \frac{M_1N_2 \times E_1P}{E_1 M_2}$ (by the similarity of the triangles $E_1 PN_1$ and $E_1 M_2 N_2$) so that the actual equation of centre is $\frac{r}{R}$ H sin $E_2 \times \frac{R}{K} = \frac{r}{R}$ H Sin E_2 where K = Mandakarna $E_1 M_2$. As M_2 moves from A_2 to P_2, the equation of centre as given by (1) gradually increases from Zero to a maximum r when $E_2 = 90°$ and decreases from this maximum to Zero when $E_2 = 180°$. Thus from what was noticed from the tabulated differences between the computed mean positions and observed true positions, the fact that those differences vanish at A_2 as well as P_2 the diametrically opposite point of the Mandoccha (not called S'ighroccha, for this word S'ighroccha will be seen to have altogether a different connotation) was verified. The maximum value of $M_2 N_2$ 'r' is termed the Antyaphala-jya or the H sine of the maximum equation of centre from which the arc could be found. In the case of the Sun and the Moon from the maximum differences between the computed and observed positions, their H sines were found and taken to be equal to 'r' in the respective cases. From this value of r, the circumferences of the circles whose radius equals r, were found and termed as Mandaparidhis. Why the circumferences were found is that in all positions of M_1M_2, the value of $M_1M_2 = r$ (in as much as the corresponding points of the two circles will be as much distant as the centres of the circles from each other so that $E_1E_2 = M_1M_2$ = Constant $= r$) so that M_2 will always lie on a circle whose centre is M_1 and radius r. This cirele is known as the Manda-NichōcchaVritta or an epicycle, where the word Nicha stands for the P_2 which is nearest the earth, and the compound word Manda-Nichōccha-Vritta means that circle which makes the planet occupy the Nicha and the Uccha points; the word Manda pertains to the Manda-phala or the equation of centre in

contradistinction to the word S'ighra which we shall shortly deal with.

In modern astronomy the equation of centre is given approximately to be equal to 2e sin m where 'm' stands for the mandakendra so that $r = 2e$. It will be shortly seen from a subsequent table that this formulation of the equation of centre gives results which closely accord with their modern values. The true or apparent positions of the Sun and the Moon could be obtained fairly well from the above formulation, so that it is stated that चन्द्रसर्यौ स्फुटौ स्यातो मान्देनैकेनकर्मणा i.e., the Moon and the Sun could be rectified by the equation of centre alone." This is quite in order for, the Sun and the Moon may be taken to be going round the Earth in ellipses, with the earth in one focus, the former relatively and the latter directly.

After having formulated the method of rectification in the case of the Sun and the Moon, the next question was with respect to the Tārā grahas i.e. Mercury, Venus and Mars, Jupiter and Saturn. As these are going round the Sun and the Sun going round the earth relatively, the process of rectification got complicated. In the first instance, the ancient astronomers must have tabulated the differences between the mean computed positions and the observed true positions. In the case of Mercury and Venus, the case appealed different from what it was in the case of the other three planets, for the simple reason that the mean positions of the former were taken to coincide with the mean Sun. This meant that for rectification, the elongation had to be computed and added to or subtracted from the mean position of the Sun to get the apparent geocentric positions of Mercury and Venus. The analogy of the method of the formulation of the Mandaphala is taken here also by imagining (1) eccentric circular motion and (2) postulating an Uccha. In the case of the Mandaphala, the equation was zero when the mean planet coincided with the Mandoccha. Here the equation is zero when

elongation is zero, i.e. when the apparent geocentric position of the planet coincides with the Sun, who is taken to be the mean planet. Naturally therefore the Uccha is taken to coincide with the Sun the mean planet, when the planet is in conjunction with the Sun. The maximum equation was had in the case of the Mandaphala when the arc between the Uccha and the mean planet was a right angle. So, here also, the maximum equation i.e. the maximum elongation should be had when the Uccha is a at right angle from the Sun. Thus an Uccha was postulated with the following criteria namely (1) It should be a point moving in a geocentric circle (2) It should coincide in direction with the Sun when the planet is in conjunction with the Sun (3) It should be removed by a right angle from the Sun when the elongation is maximum (3) It should be removed from the Sun by 180°, again when the planet coincides in direction with the Sun (4) It should have a longitude exceeding that of the Sun by 270 when again the elongation is a maximum on the other side and finally (5) It should complete a circle with respect to the Sun when again the planet coincides in direction with the Sun.

When such a point was conceived it is clear that this Uccha is not the same as the planet, as some have mis construed, because while the planet oscillates about the Sun by a particular angle (29° in the case of Mercury and 45° in that of Venus) in Uccha completes a circle with respect to the Sun and further as Hindu Astronomy postulated geocentric motion, the Uccha is a point construed as going in a geocentric circle. By the above postulation the synodic period of the Uccha is equal to the period of oscillation of the planet about the Sun. But the latter period is no other than the synodic period of the planet so that the synodic periods of the Uccha and the planet coinciding their sidereal periods should be equal. In other words the Uccha so conceived is a point other than the planet going round in a geocentric circle and having a geocentric side-

real period equal to the heliocentric sidereal period which again means that the geocentric longitude of the Uccha is the heliocentric longitude of the Planet. Thus the radius vector to the planet from the Sun is parallel to the geocentric radius vector of the Uccha. This accounts as to how the heliocentric sidereal periods of Mercury and Venus could be found under a geocentric concept and also as to how the heliocentric planets are themselves spoken of as their respective Ucchas, while their mean planet is the same as the Sun. On this count it was mentioned by the Hindu Astronomers ज्ञशुक्रयोः ग्रहः सूर्यो भवेत् तौ शीघ्रनाकौ ie. The mean Planet of Mercury and Venus is the Sun himself where as they are themselves spoken of as their Ucchas. The phrase 'they are themselves' in the above statement is significant as it connotes that the word 'they' stands for the heliocentric planets, though it was uot stated in so many words. Shortly we shall see also that the centre of the eccentric circle coincides with the centre of the Sun also and applying Bhāskara's statement 'यस्मिन्वृत्ते भ्रमति खचरो नाऽस्य मध्यं कुमध्ये' i.e. the centre of the circle in which the planet moves does not coincide with that of the Earth', the Sun was, though unwittingly taken as the centre of the Planetary motion. Thus we see how even the geocentric postulation also could help computation of the Planetary positions, the mathematics behind revealing heliocentric motion. What Copernicus achieved was that he identified that the point about which the planets revolved which was construed by the Hindu astronomers as an imaginary point not coinciding with the earth's centre, was no other than the Sun himself.

In the case of Mercury and Venus the so-called S'īghra-phala came to be discovered first and we shall presently see why their elongation was called S'īghra-phala and how the Uccha postulated as above came to be termed S'īghroccha. Since initially the equation was to be zero, when the Planet and the Uccha coincided with the Sun

and then the elongation has to increase as the Uccha gained over the Sun, the initial conjunction was the modern Superior conjunction. The other position of the Uccha when again the elongation i'e. the equation is Zero must be therefore the Inferior conjunction. Also at the motion of Superior conjunction, the planet must be having the maximum daily motion, as it is clear from a heliocentric figure that at that point the relative motion of the planet with respect to a geocentric observer is the sum of the velocities of the planet and the earth. Henee this Uccha is spoken of as the S'īghroccha also because the Uccha being a geocentrically moving point having heliocentric angular motion, its velocity is always greater than that of its planet namely the Sun. The word Uccha is applied because at the Superior conjunction the planet is farthest or highest from the earth. The excess of the longitude of this Uccha over the longitude of the planet ie. the Sun is known as the S'īghrakendra or anomaly as it is said in the verse under commentary 'चलोच्चं ग्रहोनं भवेत् शीघ्रकेन्द्रम्'. Thus in the case of Mercury and Venus, the S'īghraphala came to be discovered first. This being discovered, formulated as will be shortly shown, and applied to the mean Sun as the planet, still it was found that there was a difference between the computed position and the observed position. Such differences were tabulated. By analogy from the case of the Sun and the Moon, it was thought that there should be also a Mandoccha here also, so that the point indicated by the position of the mean planet after being corrected by the equation where the above tabulated difference was zero, was identified as the Mandoccha. From the H Sine of the maximum difference taken as the radius of the Manda epicycle, its circumference was then computed.

In the case of the superior planets, we have already said that the geocentric sidereal periods accord with their heliocentric ones. Calculating the mean position of the and finding its difference from the observed true

position, such differences were tabulated. It was discovered that these difference almost vanished when the planet was in conjunction with the Sun and attained a maximum when the elongation was nearly a right angle from an analogy from the Manda-Karma i.e. process of obtaining the Manda-phala. Since the differences attained their maximum value when the elongations were nearly a right angle it could be seen that the Sun played the part of the Uccha in this case. As the Sun has a quicker motion than the planet and also as at conjunction the planet has the quickest motion relative to the Earth while it is farthest from the Earth the Uccha ie the Sun here, is termed S'īghroccha. The excess of the longitude of the Sun over that of the planet is termed accordingly the S'īghra-kendra and the S'īghra-phala the equation was formulated as will be shown. Applying this S'īghra-phala to the mean position, the differences still found between the position so obtained and the observed true position were tabulated. The point indicated by the above position where the difference was found to be zero, was identified as the Mandoccha, and through the maximum difference, the Mandaparidhi was formulated.

In the above discourse, we have tried to give an account of how the originators of Hindu Astronomy could give us a workable system. We never assumed that an Āgama gave us the numbers of sidereal revolutions of the planets or the measures of the epicycles Manda or S'īghra. But in the explanation given by Bhāskara under Bhaganōpapatti, one will notice that when Bhāskara gave the proof of the Moon's sidereal revolutions, he said that having got the true positions of the Moon on two consecutive days, the mean positions were computed from the true by an inverse process of applying the equation of centre, and getting the mean daily motion of the Moon from those mean positions, the number of sidereal revolutions in a Kalpa were obtained. Here the Upapatti or the proof adduced by Bhāskara was not a proof but only a verification in as much as (1) he assumed the formulation

of the Mandaphala from the Āgama without pointing out how it was formulated and (2) he assumed the period of a Kalpa and that at the beginning of the Kalpa the planets were all in conjunction at the first point of Aries. Similarly in the Upapatti adduced by Bhāskara with respect to the Mandocchas of the planets, he assumed the formulation of S'īghra-phala on the basis of Āgama without proving how the concept of S'īghra-phala was arrived at by the founders of Astronomy and how the difference between the observed apparent positions and the computed mean positions, was resolved into two equations the Mandaphala and the S'īghra-phala. In the proof adduced with respect to the S'īghroccha of Mercury and Venus also Bhāskara did not mention anything as to how their heliocentric sidereal periods could be obtained but simply assumed the formulation of the Mandaphala and S'īghra-phala as already being there on the basis of Āgama.

We shall now proceed to describe the method of formulation of the S'īghra phala with respect to the five Tara-grahas, star planets namely Mercury, Venus and Mars, Jupiter, Saturn and show how so different a set of geometries of the ancients and the moderns the one geocentric and the other heliocentric could give identical formulation with respect to S'īghra-phala. Let us consider the case of Mercury and Venus in the first instance.

Having taken the mean Sun to play the part of the 'Graha' in the case of Mercury and Venus and having formed a concept of S'īghroccha as mentioned before, whose geocentric period of revolution was determined, without suspecting it to be the heliocentric sidereal period of the planet the ancient Hindu astronomers assumed by analogy from the case of Mandaphala with respect to the Sun and Moon, that the centre of the circle in which these planets revolve does not coincide with the centre of the Earth. In other words, they continued eccentric circle theory here so that without suspecting heliocentric

their mathematics led to them to make the centre of the eccentric circle coincide with the Sun himself. On this basis alone it was given to Copernicus to formulate heliocentric theory, sponsored by a thought that the Heavenly Sun could not be deemed as a satellite of the 'Mundane' Earth.

(Ref. fig. 9). The same figure 9 will also serve the purpose to obtain the S'ighraphala, only M_1 M_2 will be now on the right hand side of the S'ighrocchas A_1 A_2, for, the latter will be taken to be in advance of the mean planets Kakshā-Vrittīya Madhyagraha M_1 (i.e. mean planet of the deferent) and prati-Vrittīya Madhyagraha M_2 (i.e. mean planet of the eccentric). The points A_1 add A_2 are themselves called the Kaksha-Vrittīya S'ighrōccha and prati-Vrittīya S'ighrōccha respectively. As was shown in the case of the Mandaphala from the eccentric figure 9, M_2 N_2 = $\frac{r}{R}$ H sin (Kendra) so that $PN_1 = \frac{r}{R}$ H sin (Kendra) $\times \frac{R}{K}$ $= \frac{r}{K}$ H sin (Kendra) $= \frac{\text{Antyaphalajya} \times \text{Sīghrakendrajyā}}{\text{S'ighrakarna}}$.
We shall take this for elucidation in the appropriate context.

Verse 19. Three Rasis each of 30° constitute a quadrant, and there are four quadrants in a circle which are respectively odd, even, odd and even. In the odd quadrants the Kendra covered is itself called Bhuja whereas in the even ones, the complement thereof is called Bhuja. Also, the complement of the Bhuja is called the Koti.

Verse 20. R − H sine = Co. H versine and R − H cosine = H versine and R — Co. H versine = H sine, R — H versine = H cosine.

Verse 21. Also $\sqrt{R^2 - \text{H sine}^2}$ = H cosine,
= H sine. Similarly
Dyujya, $\sqrt{R^2 - \text{Dyujya}^2}$ = Krāntijya; $\sqrt{R^2 - \text{Drig-jyā}^2}$

$=$ S'anku, $\sqrt{R^2 - \text{S'anku}^2}$ $=$ Drig-jyā. In all the cases cited above, the radins R happens to be the hypotenuse.

Comm. The convention in verse 19 corresponds to saying in modern trigonometry that $\sin \overline{90 + \theta} = \cos \theta$ $\sin (180 - \theta)$ $\sin \theta$, $\sin \overline{180 + \theta} = - \sin \theta$, $\sin \overline{270 - \theta} = - \cos \theta$ $\sin \overline{270 + \theta} = - \cos \theta$, $\sin \overline{360 - \theta} = - \sin \theta$. In Hindu trigonometry the sign is understood and not explicitty mentioned.

Krāntijyā, Dyujyā, Drig-jyā and S'anku, are respectively H sin δ, H cos δ, H sin Z, H cos Z where δ is declination and Z the Zenith-distance of a celestial body. Taking the radius of the celertial equator to be R, the radius of the diurnal circle of a celestial would be equal to R $\times$ cos δ $=$ H cos δ which is called Dyujyā because it is the radius of the diurnal circle.

Verse 22. The lengths of the circumferences of the Manda—epicycles are respectively 13° - 40′, 31° - 36′, 70°, 38°, 33°, 50,* for the Sun, Moon, Mars, Mercury, Jupiter, Venus and Saturn.

Comm. Bhāskara has given these measures reportedly on the basis of Āgama or ancient authority. The peculiarity of measuring the circumferences in degrees less than 360°, is due to the idea that these circumstances are measured in relation to that of the deferent or Kakshā Vritta taken to be 360°. In other words, circumference of the epicycle of a planet as given above : circumference of the mean orbit : : x : 360 $=$ radius of the epicycle : radius of the mean orbit where x is the measure of the circumference of the planetary epicycle. It may be mentioned once again that the radius of a planetary epicycle is the measure of the greatest

* The printed book of Brahma Sphuta Siddhānta gives in the case of Saturn 30° only which might have been the mistake of the scribe (Vide verse 36, Spashtādhikāra B. S.). In the place of शून्यरामाः it ought to have been

equation of centre pertaining to the planet which may be taken to be equal to $2e$ as a first approximation where e is the eccentricity of the elliptic orbit of the planet.

It may be further mentioned here that in Sūryasiddhānta, as well as elsewhere in this work, the circumferences are given to vary continuously. This variability curiously achieves ellipticity in the orbit as may be seen as follows.

In the case of the Sun, the epicycle has a periphery of $14°$ when $m = 0$ or $180°$ and of $13\frac{2}{3}°$ when $m = 90°$ or 270 according to Sūryasiddhānta, where m is the Mandakendra or mean anomaly. At any arbitrary point, where

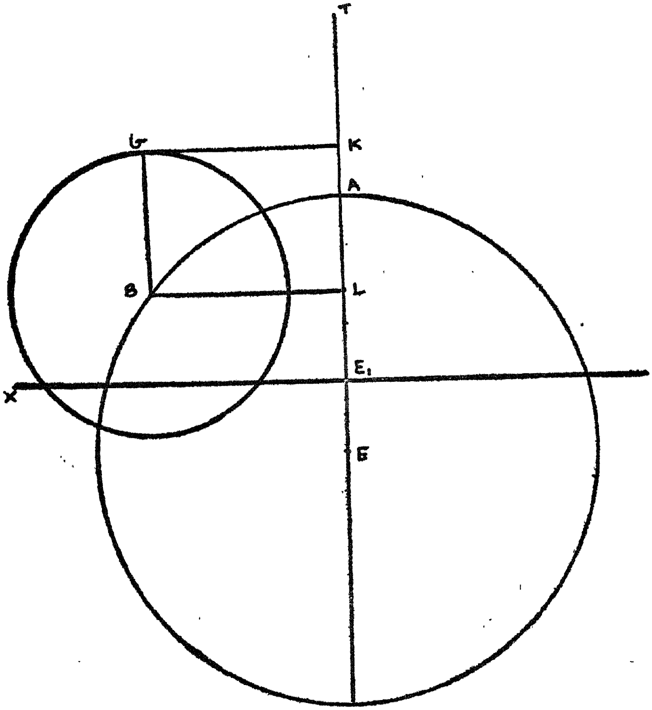


Fig. 10

the mean anomaly is m the periphery is given to be $14° - \frac{20' \text{ H sin m.}}{R}$ The corresponding radius will therefore be $r - \frac{20.}{2\pi} \cdot \frac{\text{H sin m}}{R} = r - \lambda \text{ H sin m}$ (say) where $r = \frac{14°.}{2\pi}$ (Ref. fig. 10) Let $\mathcal{E}$ be the earth's centre, A the position of the apogee EE_1 = the radius of the epicycle measured along $\mathcal{E}$A, B any arbitrary position of the mean planet and b the position of the planet in the epicycle. Here the radius Bb is not equal to the max. radius equal to EE_1 i.e. r but equal to r − H sin m. Take E_1 as the origin and E_1A as the y—axis and a perpendicular to E_1A through E_1 namely E_1x as the positive direction of the X-axis. If the mean anomaly BEA be m, then the coordinates of the true planet are given by x = BL = H sin m (1) y = E_1L + Bb = EL − EE_1 + r − λ H sin m = H cos m − r + (r − λ H sin m = H cos m − λx.

$\therefore$ y + λx = H cos m (2) Squaring and adding I and II $x^2 + (y + \lambda x)^2 = \text{H sin}^2\text{m} + \text{H cos}^2\text{m} = R^2$

$\therefore$ $x^2(1+\lambda^2) + 2\lambda xy + y^2 = R^2$ which is an ellipse with centre E_1

Verses 23, 24, 25. The peripheries of S'ighra epicycles. The peripheries of the epicycles of the star-planets Mars, Mercury, Jupiter, Venus and Saturn are respectively 243°-40′, 132°, 68°, 258° and 40°. The H sine of the Manda mean anomaly of Venus being multiplied by 2 and divided by 343, and the result being subtracted from the periphery gives the rectified Manda periphery. The H sine of its S'īghra mean anomaly being multiplied by 5 and divided by R, and the result being added to the S'ighra periphery gives the rectified periphery. The smaller of the H sine or H cosine of the S'ighra anomaly of Mars being multiplied by $6\frac{2}{3}$ and divided by H sin 45° and the result in degrees being subtracted from or added to as the case may be,

the aphelion gives the rectified aphelion. The S'īghra periphery being reduced by the above degrees gives the rectified S'īghra periphery in case the S'īghra anomaly is $90° < m < 180$ or $270° < m < 360°$.

Comm. In the commentary Bhāskara adds that in the case of Venus the Mandaperiphery of 11° as given is at the end of even quadrants whereas at the end of odd quadrants it is 9°, wherefore the enunciated rectification. Similarly in the case of his S'īghraphala, the periphery of 245° mentioned is at the end of even quadrants whereas at the end of odd quadrants it is 263°, and so the suggested rectification. Again in the case of Mars, the aphelion as computed is the same at the end of all quadrants whereas in the middle of the quadrants it is to be increased or decreased by $6\frac{2}{3}°$ when the anomaly is as stated. Also in the case of this Mars, the S'īghra periphery mentioned is at the ends of quadrants. In the middle of the quadrants the periphery is to be reduced as suggested. In all these interpolations Bhāskara accepts the Āgama as enunciated by Brahmagupta.

We shall deal with the geometrical nature of these S'īghra peripheries shortly in the appropriate place.

Verse 26. To obtain what are called Bhujaphala and Kotiphala both in the case of Mandaphala as well as S'īghraphala. The H sine and H cosine of the Manda or S'īghra anomalies multiplied by the respective peripheries and divided by 360°, or multiplied by r and divided by R gives the Bhujaphala or Kotiphala where r and R are respectively the radius of the Manda or S'īghra peripheries and R the radius of the deferent taken to be 3438′. If the radius 3438′ be respectively multiplied by the Manda or S'īghra peripheries and divided by 360°, the result will be the H sine of the maximum Mandaphala or S'īghraphala, known as in either case.

Comm. As per the formulation.

$$\text{Bhujaphala} = \frac{\text{H sin m} \times c}{360} = \frac{\text{H sin m} \times r}{R}$$

$$\text{Kotiphala} = \frac{\text{H cos m} \times c}{360} = \frac{\text{H cos m} \times r}{R}$$

in the case of Mandaphala or S'īghraphala where c = periphery of the Manda or S'īghra periphery, r=Antyaphalajyā defined above R = 3438′ and m stands for the Manda or S'īghra anomaly. These Bhujaphala and Kotiphala will be used in their respective contexts.

Verses 27, 28, 29. Calculation of what is known as S'īghrakarṇa.

$$(\text{H Cos m} \pm r)^2 + \text{H Sin}^2 \text{ m} = K^2 \quad (1)$$

$$(R \pm \text{Kotiphala})^2 + \text{Bhujaphala}^2 = K^2 \quad (2)$$

$$R^2 + r^2 \pm 2\,R \times \text{Kotiphala} = K^2 \quad (3)$$

$$R^2 + r^2 \pm 2\,r \times \text{H Cos m} = K^2 \quad (4)$$

The arc of the H Sine of the equation of centre is called the Mandaphala.

Comm. Ref. fig. 9. From triangle $E_1M_1M_2$ $(E_1M_1 + M_1N_2)^2 + M_1N_1^2 = E_1M_2^2 = K^2$. But M_1N_2 = Kotiphala and M_2N_2 = Bhujaphala defined previously so that we have the second formula for K enunciated above. From the similarity of the triangles $E_1M_1Q_1$ and $M_1M_2N_2$ we have $\frac{M_2N_2}{M_1Q_1} = \frac{M_1M_2}{E_1M_1}$ so that $M_2N_2 = \frac{r}{R} \times M_1Q_1$; Since M_1Q_1 is called the Bhuja, the corresponding M_2N_2 in the Antyaphalajyā triangle is called the Bhujaphala. Similarly M_1N_2 is called the Kotiphala.

Again $(E_1Q_1 + Q_1Q_2)^2 + M_2Q_2^2 = E_1M_2^2$ where E_1Q_1 = H Cos m, $Q_1Q_2 = M_1N_2 = r$ and $M_2Q_2 = M_1Q_1$=H Sin m. From this we have the first formula enunciated.

Again expanding $(E_1M_1 + M_1N_2)^2$, $(E_1M_1 + M_1N_2)^2 + M_2N_1^2 = E_1M_2^2$ we have the third formula; similarly expanding $(E_1Q_1 + Q_1Q_2)^2$, $(E_1Q_1 + Q_1Q_2)^2 + M_2Q_2^2 = E_1M_2^2$ we have the fourth formula.

Since the S'ighraphala has been defined to be equal to $\frac{r}{K}$ H Sin m, we have had the necessity of knowing the value of K. The convention of signs mentioned in the formulation in the words 'योगो मृगादावथ कर्कटादौ केन्द्रेऽन्तरम्' is due to the fact that cosine is positive in the fourth and first Quadrants and that the Kotiphala becomes negative in the 2nd and 3rd Quadrants as could be seen by drawing the figure in those Quadrants.

Now we shall prove what is most important, namely that postulating an entirely different geocentric motion how the Hindu Astronomers could formulate the S'ighraphala which accords exactly with the heliocentric theory, assuming of course coplanar circular orbits. Let figures 11 and 12 pertain to the modern heliocentric geometry,

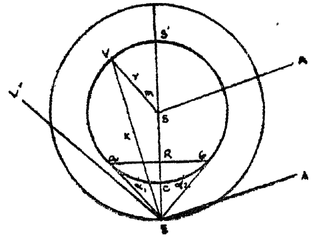


Fig. 11

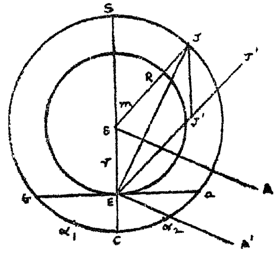
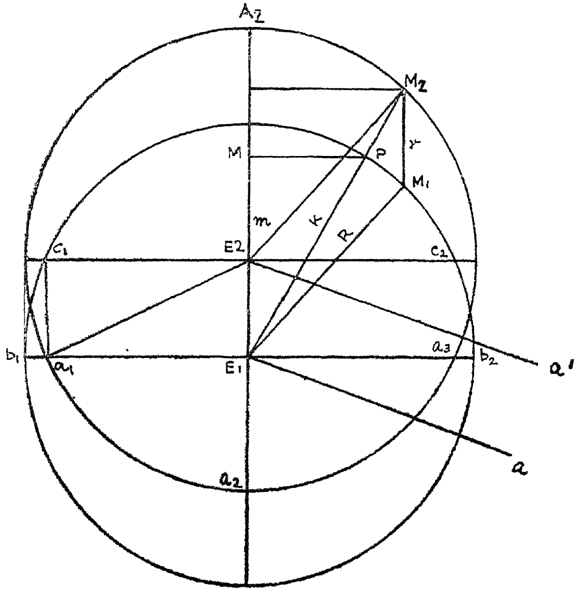


Fig. 12

the former with respect to the Inferior planets Mercury and Venus signified by V, and the latter to the superior planets Mars, Jupiter and Saturn signified by J. Let fig. 13 pertain to the Hindu geocentric geometry dealing with both the Inferior and Superior planets as well. In the heliocentric figures let S=Sun, E=Earth, SA=directon to ii, the Zero-point of the Zodiac from the Sun,



geocentric direction towards Aswini. Let SV, EV^1, be the heliocentric and geocentric directions of the S'īghroccha where V is the actual planet and V^1 an imaginary point. Draw EJ' 11 to SJ. Let the radius of the inner and outer heliocentric circles be respectively r and R. Let K be the radius vector to the planet in both the figures.

Let in fig. 13, E_1=Earth's centre, E_2=the centre of the eccentric circle which we shall presently show to be coinciding with the Sun's position. Let M_1, M_2 represent the mean planets in the deferent and the eccentric known as Kaksha-Vrittīya Madhyagraha and prati-Vrittīya Madhyagraha. Let A_1, A_2 be the S'īghrocchas in the deferent and the eccentric. Let $E_1E_2 = r$ known as Antyaphalajyā and R the radius of both the deferent and the eccentric. Let K be the radius vector to the planet known as S'īghra-Karṇa.

We shall prove that m the S'ighra anomaly of the Hindu figure will be the same as 'm' as marked in the heliocentric figures. S'ighra anomaly is defined as longitude of S'ighroccha—longitude of the $A_1 - aE_1M_2 = a^1\widehat{E_2}A_2$ $a^1\widehat{E_2}M_2 = m$ fig. 13. In the heliocentric fig. 11, since A^1EV^1 is the longitude of the S'ighroccha, and $A^1\widehat{E}S$ the longitude of the Sun treated as the Madhyagraha of the Inferior planet $A^1\widehat{E}V^1 - A^1\widehat{E}S = V^1\widehat{E}S = V\widehat{S}S^1 = m =$ the S'ighra anomaly. In fig 12, $m = S^1\widehat{S}J = S\widehat{E}J = A^1\widehat{E}S - A^1EJ^1 = A^1ES - A\widehat{S}J$ = Longitude of the Sun treated as the S'ighroccha of the Superior planet minus longitude of the heliocentric planet known as Mandasphutagraha or the planet rectified for the Mandaphala or equation of centre=S'ighra anomaly In the case of the Inferior planet the heliocentric direction of the planet is equal to the S'ighroccha. Now consider the triangles ESV, JSE, $E_1M_1M_2$ of the three figures. Evidently $E\widehat{S}V = J\widehat{S}E = E_1\widehat{M_1}M_2 = 180 - m$ = Supplement of S'ighra anomaly. If, further it is shown and (it will be shown subsequently) $\frac{SV}{SE} = \frac{SE}{SJ} = M_1M_2$ the similarity of the triangle E_1M_1M geocentric figure separately with ESV and JSV will have been established. Taking this similarity to have been established, $M_1\widehat{E}M_2$ known as S'ighraphala will be equal to $S\widehat{E}V$ in fig. 11 and $S\widehat{J}E$ in fig. 12. In fig. 13, M_2 the prativritta Madhagraha is also known as the pāramārthikagraha or the actual planet where as p its geocentric position on the Kakshāmandala is taken to be the true planet or apparent position of the planet. In figures 11 and 12, EV and EJ are the directions to the true planets V and J so that the angles between the Sphutagraha and the Madhyagraha (ie the Manda Sphutagraha =

(in fig 11) and $= A^1EJ$ - Śīghraphala. Once the similarity of the triangles fig. 12 is established, the equality of the Śīghraphala will be established. Also due to the similarity mentioned above the formulae for K as given in Hindu Astronomy should also accord with that in the heliocentric figures. In fact in the heliocentric figures $K^2 = R^2 + r^2 + 2Rr \cos m = R^2 + r^2 + 2RH \cos m$ which is indentical with the four formulae given before as per verses 27, 28, 29. It will be seen that the epicycle (M_1) with radius M_1M_2 will be identical with the inner circles in the heliocentric circles, whereas the Kakshamandal (E_1) with radius R will be identical with the outer circles of the heliocentric figures.

Before we proceed further, we shall annex the table wherein the ratio r/R as given in Hindu Astronomy will be seen to accord with that in modern astronomy.

Planet	Periphery of the Śīghra-epicycle	Periphery of the deferent	Ratio	Value in modern astronomy taking Earth's radius to be unity
Mercury	132°	360°	$\frac{132}{360} = \cdot37$	·387
Venus	258°	360°	·716	·723
Mars	$243\frac{2}{3}$°	360°	1·5	1·52
Jupiter	68°	360°	5·3	5·2
Saturn	40°	360°	9	9·5

In the light of this table the similarity of the triangles ESV_1 and JSE with $E_1M_1M_2$ is now established.

Formula for S'ighraphala from the heliocentric figures

In fig. 11, $\frac{r}{K} = \frac{\sin SEV}{\sin ESV}$ so

$$\frac{r}{K} \times \sin m$$

In fig. 12 $\frac{r}{K} = \frac{\sin \widehat{SJE}}{\sin \widehat{ESJ}}$ so that $\sin \widehat{SJE} = \frac{r}{K} \sin m$

Both these accord with the Hindu formula.

It will be interesting to point out here that in fig 11, keeping the earth constant and supposing the Sun S to go in a circle with centre E and radius ES, the orbit of the Inferior planet V will play the part of the epicycle of Hindu Astronomy. Thus in the case of the Inferior planets, the epicyclic theory is only a different version of the heliocentric theory. In the case of the Superior planets, however, (fig. 12) cut off $EJ^1 = SJ$ along EJ'' parallel to SJ; then J^1J will be parallel to ES just as M_1M_2 is parallel to E_1E_2 in fig. 13. Then the circle with E as centre and EJ^1 as radius corresponds to the deferent of fig 13, whereas the circle (J^1) with centre J^1 and radius J^1J corresponds to the epicycle. The circle with S as centre and radius SJ corresponds to the eccentric.

Verse 30. The equation of centre pertaining to the Sun, and the Moon using a simpler table of H sines where the radius = 120 units. The H sines of the mean anomaly as found from the simpler H sine table where radius = 120, multiplied by 20, and divided by 1103 and 477 respectively gives the equation of centre of the Sun and the Moon in degrees.

Comm. The maximum equation of centre with respect to the Sun is 2°-10′-31″. Then the argument is " If by the H sine of the anomaly equal to the radius 120, we have the above max. equation what shall we have for H sin m ?".

The answer is $\frac{\text{H sin m} \times 2°\text{-}10'\text{-}31''}{120} = \frac{2\frac{21}{120}°}{120} \times \text{H sin m}$

very approximately $= \frac{261}{14400}$ H sin m $= \frac{20}{1103}$ H sin m

Similarly in the case of the Moon, the maximum equation of centre is 5°-2′-8″. By the same argument as above we have $\frac{\text{H sin m} \times 1133}{225 \times 120} = \frac{\text{H sin m} \times 20}{\frac{54000}{1133}} = \frac{\text{H sin m} \times 20}{477}$

Verse 31. Rectification of the mean daily motion of the Sun and the Moon.

The H cosine of the mean anomaly divided by 54 in the case of the Sun and in the case of the Moon multiplied by 4 and divided by 7 gives the increment or decrement in the respective mean motions according as $90 < m < 270$ or $270 < m < 360 + 90$.

Comm. We have Equation of centre $= \frac{r}{R}$ H sin m $=$ E (say) so that differentiating $\delta E = \frac{r}{R}$ H cosm $\frac{\delta m}{R}$. But $\frac{r}{R}$ H cosm is called kotiphala and δm is called Kendra gati so that $\delta E = \frac{\text{Kotiphala} \times \text{Kendragati.}}{R}$ Since Kotiphala is negative when $90 < m < 270$ δE is negative but in Hindu Astronomy we measure the Kendra not from perigee as in modern astronomy but from aphelion so that the equation of centre is strictly $- \frac{r}{R}$ H sin m if sign is also taken into consideration. Hence δE must be + ve. Since $M + E = S$ where M is the mean planet, E the equation of centre and S the true planet $\delta S = \delta m + \delta E$ so that the true motion is equal to the mean motion plus δE. As δE is +ve when $90 < m < 270$ as mentioned above we have to add this to the mean motion to get the true motion. This

is called gatiphala or what is to be added to the mean motion to give the true motion. Incidentally we have commented on the contents of verse 37.

Verse 32. The S'īghra phala with respect to the Star-planets.

$\frac{\text{H sin m} \times r}{\text{R}}$ being multiplied by the radius R or the product of H sin m and r being divided by K the arc of the result gives the S'īghraphala.

Comm. In the verse it is mentioned that the product of the Bhujaphala and the radius is divided by K so that the formula for the Bhujaphala being $\frac{\text{H sin m} \times r}{\text{R}}$ the S'īghra-phalajya will be $\frac{\text{H sin m} \times r}{\text{K}}$ which is stated in the alternative. We have already derived this formula before where we got H sin $E_2 = \frac{\text{H sin m} \times r}{\text{K}}$. The arc of this will be E_2 i.e. the S'īghra-phala, (E_2 because we take E_1 as the Mandaphala).

Verses 33 and 34. An alternative formulation of S'īghra-phala.

H sin m being multiplied by R and divided by K and the difference between the arc of the result and H sin m will be the S'īghra-phala. Here H sin m belongs to the eccentric. The arc of the maximum S'īghra-phalajya added to or subtracted from 90° will give respectively the Quadrants and the H sine will have to be taken of the elapsed Kendra or its Koti according as the Quadrant is odd or even.

Comm. Ref. fig. 13. From the similarity of the triangles E_1M_2N and E_1PM,

$$= \frac{\text{R}}{\text{K}} \times M_2N$$

But PM is the H sine of m^1 where m^1 is called Sphuta kendra (m is called the madhya-Kendra). Hence

$$H \sin m^1 = \frac{R}{K} \times M_2N = \frac{R}{K} \times \text{Bhujajyā (in the eccentric)}$$

$$\therefore\ m^1 = H\,\text{Sin}^{-1}\left(\frac{R}{K} \times \text{Bhujajyā}\right) = PA_1$$

$\therefore$ M_1P = S'īghraphala = $M_1A_1 - PA_1$ = Kakshya-mandala Bahu minus the chāpa m^1 — Here a clear understanding of the word Bāhu or what is the same Bhuja should be had. The arc pertaining to the angle m in the eccentric is known as the Bāhu in the eccentric and that to the angle m in the deferent as the Bāhu in the deferent. When $0<m<90$, m is itself spoken of as Bāhu; when $90<m<180$, $180-m$ is spoken of as the Bāhu; when $180<m<270$, $m-180$ is the Bāhu and when $270<m<360$, $360-m$ is the Bāhu. Thus the Bāhu is that angle whose H sine will be H sin m numerically. When it is said in the verse 'त्रिज्याहता कर्णहृता भुजज्या' the word Bhuja is the arc M_2A_2 as is mentioned in the same verse 'ज्ञेयोऽत्र बाहुः प्रतिमण्डलस्य'. The second part of the verse divides the eccentric circle into such quadrants that in them S'īghraphala increases from Zero to a maximum, decreases again from a maximum to zero, again in creases from zero to a maximum and again decreases from a maximum to zero. Thus at A_2 of the eccentric the S'īghraphala is zero; at a_1, it is a maximum namely the arc b_1c_1 where H sin $b_1c_1 = a_1c_1 = r$; Thus in the course of A_2a_1 the arc of the eccentric the S'īghraphala gradually increases from zero to a maximum and in the course of a_1a_2 the S'īghraphala decreases from a max to zero. Again from a_2 to a_3 it increases from zero to a max and from a_3 to A_2 it decreases from the maximum to zero. Thus the quadrants in the case of S'īghraphala are not of 90° but arcs A_2a_1, a_1a_2, a_2a_3 and a_3A_2 which are respectively of magnitude $90° + H\,\text{Sin}^{-1}r$, $90 - H\,\text{Sin}^{-1}r$, $90 - H\,\text{Sin}^{-1}r$ and $90 + H\,\text{Sin}^{-1}r$. In the case of Mandaphala also, the quadrants should have been of the same magnitude if the so-called Karnānupāta has been postulated i.e. reducing the Manda-

phala from the extremity of the Karna to the extremity of the radius in the deferent; but as this Karnānupāta is not adopted, the difference being negligible the quadrants are all of equal magnitude i.e. each of 90°.

In the course of the commentary of this verse Bhāskara mentions that for Mercury, as the maximum S'īghraphala is 21°-31′-43″, the quadrants are of magnitude 3-21-31-43, 2-8-28-17, 2-8-28-17 and 3-21-31-43 respectively.

Also in the commentary Bhāskara adds that a_1 which is the point of intersectiou of the eccentric with the horizontal diameter E_1b_1 the S'īghraphala is maximum and that at that point the mean motion is itself the true motion

गतिः स्पष्टा परं

. That the S'īghraphala at a_1 and a_3 is maximum is clear from the figure 13, where it is equal to the arcs b_1c_1 and b_2c_2 whose H sine is equal to r. To prove that the mean motion is itself the true motion, we have the equation $M_2 + E_2 = S$ where M_2 is the mean planet here or the Mandasphutagraha or planet rectified for the equation of centre, (by the equation $M_1+E_1=M_2$, M_1 being the original mean planet and E_1 the equation of centre) so that $\delta M_2+\delta E_2 = \delta S$ where δM_2 is the mean motion here, δS the true motion and δM_2 is the variation in the S'īghraphala; but at a_1 and a_3 E_2 the S'īghraphala being maximum δE_2 is zero. Hence $\delta M_2 = \delta S$ which means that the mean motion is itself the true motion.

Verse 34. Latter half, 35 and 36 former half.

The mean planet rectified for the equation of centre or Manda-phala is called Mandasphuta. Then subtracting the longitude of the Mandasphuta from that of the respective S'īghroccha, the result will be the S'īghra anomaly from which the S'īghra-phala is to be obtained. Rectifying the Mandasphuta for this second equation namely S'īghraphala, again obtaining therefrom the equation of centre effecting

this in the original mean planet and again correcting for S'īghraphala and repeating the process till a constant value is obtained, the true planet is had with respect to the star-planets other than Mars. But with respect to Mars, let first the mean planet be corrected for half of the equation of centre. Then make half of the correction of S'īghraphala. Take the resulting planet to be the mean planet and again finding the equation of centre, make this whole correction in the original mean planet. Again taking the resulting planet to be the Mandasphuta effect the entire S'īghraphala. Then we have the true planet.

Comm. The Suryasiddhānta stipulates the same kind of correction in the case of all the star-planets.

ईं

शैघ्रयं फलार्धं प्रथमं ततो

ग्रहे ॥" i.e. In the first place half of the S'īghraphala is to be effected in the mean planet; taking that to be the mean planet and computing the equation of centre half of it is administered; taking the resulting to be the mean planet and computing the equation of centre, the entire equation of centre is now to be administered in the original mean planet; taking the result to be the Mandasphutagraha, and computing the S'īghraphala, it is to be administered in full in the Mandasphuta. Then we have the true planet.

In modern astronomy, the equation of centre is first done and the result will be the planet in its heliocentric elliptic orbit. To reduce it to the geocentric position, the second correction is made which corresponds to the Hindu S'īghraphala. Thus the two corrections administered successively gives the apparent or true geocentric planet.

Though the mutual relationship as conceived between the equation of centre and the S'īghraphala in Hindu Astronomy is deemed irrational by modern interpreters of Hindu tronomy, there is some rationale in the process as ex-

plained by this author in his work 'A critical study of Ancient Hindu Astronomy' (published by the Karnatak University) page 98.

Verse. Cited from Golādhyāya.

The equation of centre is to be applied to the mean planet to obtain the centre of the S'īghraepicycle; then to obtain the true position the S'īghraphala is to applied to the Mandasphutagraha; hence the two equations are mutually related so that the true position is obtained after repeated application of the two equations.

Comm. Explained above.

Verse 36 latter half and 37. The true daily motion of the planet is the excess of the longitude of the true planet of the next day over that of the true planet of the previous day.

The Kotiphala being multiplied by the daily motion of the Manda mean anomaly and divided by the radius, and the result being added to or subtracted from the mean motion, gives what is called Mandasphutagati.

Comm. Already explained before. If M_1, M_2 and S be the mean planet, Mandasphutagraha, and the true planet respectively, and if E_1 and E_2 be respectively the two equations, then

$M_1+E_1=M_2$, $M_2+E_2=S$ so that

Here δM_1=mean daily motion, δE_1=daily variation in the Mandaphala, δM_2=daily motion of the Mandasphuta graha or what is the same Mandasphutagati, δE_2=daily variation in the second equation and δS = True daily motion.

Lallāchārya formulates the Mandasphutagati in a different manner which is equally correct (Ref. verse 45 Spaṣṭādhikāra Siṣyadhī Vriddhida "त्रिज्याहता ग्रहगतिः मृदुकर्णभक्ता मन्दस्फुटा भवति" i. e. $\frac{\delta m \times R}{K}$ = Mandasphutagati where δm = Madhyagati or mean motion and K is the Manda Karṇa equal to $\sqrt{R^2 + r^2 \pm 2 R r \cos m}$

∴ Mandasphutagati =

$$\frac{\delta m \times R}{\sqrt{1 \pm 2 \frac{r}{R} \cos m + \frac{r^2}{R^2}}} = \delta m \left(1 \pm \frac{2r}{R} \cos m \right)^{-\frac{1}{2}}$$

$= \delta m \left(1 \mp \frac{r}{R} \cos m \right)$ neglecting the smaller term $-\frac{r^2}{2R^2}$ within brackets

$$= \delta m \mp \frac{r \text{ H} \cos m \, \delta m}{R^2} \delta m = \delta m \mp \frac{r \text{ H} \cos m}{R} \times \frac{\delta m}{R}$$

$= \delta m \mp \frac{\text{Kotiphala} \times \text{Mandakendragati}}{}$ as given by Bhāskara.

Verse 38. In the case of the Moon, obtaining the true Moon for a particular moment and his daily motion for the day, the ending moment of the tithi near at hand is to be computed with that daily motion, and the method of successive approximations is to be used to rectify the ending moment. In the case of the ending moment of the tithi being sufficiently far away, then it does not matter even if the above daily motion is applied to get the approximate ending moment. In as much as the Moon's daily motion is great and varies from moment to moment the motion at the moment is to be used.

Comm. Strictly speaking the ending moment of every tithi is to be computed by the method of successive approximation. That is why in the computation of eclipses,

Moon's hourly motion is given in modern almanacs. Since it is very cumbrous to use the method of successive approximation to determine the ending moment of every tithi, the Hindu almanac-makers generally compute the ending moment of a tithi using the daily motion of the moon computed for the moment of Sun-rise of the day. Only for ritual purposes, the method of successive approximation is used and also in the computation of eclipses.

Verse 39. Computation of the S'īghragatiphala.

$$\frac{H\sin(90-E_2)\times\delta m}{K} = \delta S$$ where δl is the daily motion of the S'īghrōccha, E_2 = S'īghraphala, δm = daily motion in the S'īghra mean anomaly, K = S'īghrakarṇa, and δS the true motion of the planet. If δS is negative the planet is retrograde.

Comm. This is mathematically an important verse, and the proof given by Bhāskara really reflects his genius. Before we attend to his proof, we shall give this a modern treatment. (Ref. figs. 14, 15). SA and EA[1] are the

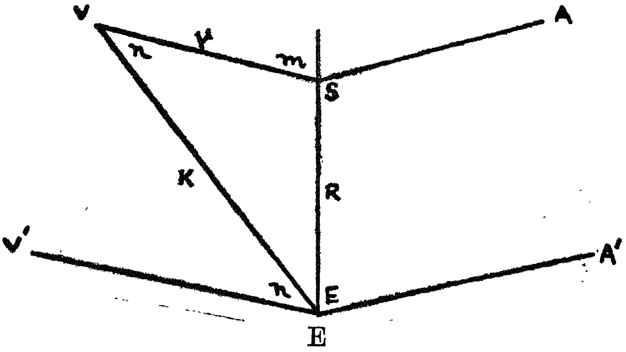


Fig. 14

heliocentric and geocentric directions to Aswini the Hindu Zero-point of the Zodiac; S = Sun, E = Earth; V = Inferior planet Venus or Mercury; J = Superior planet; E = S'īghra-phala, K = S'īghra Karṇa; M = S'īghra anomaly.

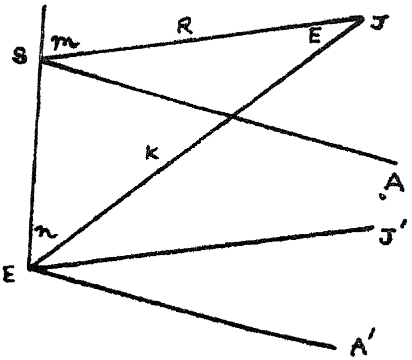


Fig. 15

True motion of the planets = δ (A'EV) or δ (A'EJ)

But δ (A'EV) = δ (A'EV'-$\hat{n}$) and δ (A'EJ) = δ (A'ES-$\hat{n}$)

In the case of the Inferior planet δ (A'EV') = δ (ASV) = S'īghrōcchagati and δn is Sphutakendragati where n is called Sphutakendra, m being called Madhyakendra. In the case of the Superior planet δ (A'ES) = S'īghrōcchagati because the Sun plays the part of S'īghrōccha in the case of a Superior planet and δn = Sphutakendragati as before. Hence in both the cases, Sphutagati=S'īghragati—Sphutakendragati. We have now to find Sphutakendragati to obtain Sphutagati, as Bhāskara remarks rightly "महत्मतिम-

रेव स्पष्टीकृता". In other words we have to find δn. From the figures K cosn − R cos m = r (i) Differentiating this we have − K sin n δn + cos n δK + R sin m δm = 0 (ii). But $K^2 = R^2 + r^2 + 2Rr \cos m$ so that $2\delta K \times K = -2Rr \sin m \delta m$ (3) Eliminating δK between (2) and

(3) − K sin nδ n − + R sin

i.e. $K \sin n\delta n = R \sin m\delta m \left(1 - \frac{r}{K} \cos n \right)$

$- \frac{R \sin}{K} \quad (K - r \cos$

But $K - r \cos n = R \cos E$.

$$\therefore \quad \delta n = \frac{R \sin m \delta m \times R \cos E}{K^2 \cdot n} \quad \text{But } R \sin m = K \sin n$$

Cancelling $\delta n = \frac{R \cos E \delta m}{K} = \frac{H \cos E \, \delta m}{K}$ as given by Bhāskara. In the formula $H \sin E = \frac{r \, H \sin m}{K}$, Bhāskara perceived the variability of both H sin m and K on the right hand side and he exclaims "न हि केन्द्रगतिजमेव फलयोरन्तरं स्यात्, किन्त्वन्यथाऽपि अद्यतनभुजफलश्वस्तनभुजफलान्तरे त्रिज्यागुणे अद्यतनकर्णहृते यादृशं फलं न तादृशं श्वस्तनकर्णहृते, स्वल्पान्तरेऽपि कर्णे भाज्यस्य बहुत्वात् बह्वन्तरं स्यादित्येतदानयनं हित्वा अन्यत् महामतिमद्भिः कल्पितम्, तद्यथा केन्द्रगतिरेव स्पष्टीकृता" i.e. "The variation in the S'īghraphala is not entirely constituted by the variation in m but also by that in K......So leaving the method of seeking δE through the formula $H \sin E = \frac{r \, H \sin m}{K}$ the great intelligent astronomers used the formula.

Sphutabhukti = S'īghra Bhukti — Sphuta Kendrabhukti (Bhukti means gati) wherein it was sought to obtain the variation in Sphutakendra i.e. $\hat{n}$ in the figures.

We have given a proof of Bhāskara's formula, which circumvented finding S'īghragatiphala but which sought directly Sphutabhukti, by using the modern heliocentric figures. We shall now see how Bhāskara could deal with such a tough problem. Refer fig 16. Let P_1, P_2 be two positions of the planet on two consecutive days relative to the S'īghra A_2, so that $P_1E_1P_2$ is the Sphuta Kendragati spoken of. It will be noted that it is not Sphutagati because P_1, P_2 are positions of the planet relative to A_2 which is itself moving (as rightly remarked by Bhāskara). It is Sphuta Kendragati because $A_2E_1P_1$ and $A_2E_1P_2$ are the

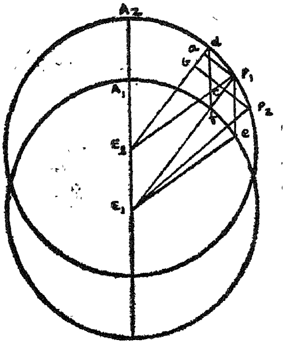


Fig. 16

Kendras on two consecutive days whereas Madhyakendragati is $P_1\hat{E}_2P_2$. Also, we have the equation.

S'ighra — Sphutagraha = Sphutakendra so that S'ighragati — Sphutagati = Sphutakendragati. Hence Sphutagati = S'ighragati — Sphutakendragati. So we have now to seek the value of $P_1\hat{E}_1P_2$. Let P_1a stand for the S'ighraphala of the first day which is equal to e f, f being the true planet of the first day. E_2 d will be parallel to E_1 f because P_1 e being parallel to E_1 E_2 and e f being equal to P_1 d, d f 11 P_1 e (parallels to E_1 E_2 cut off equal arcs on the two circles). This may be seen also as follows. Since P_1 d is taken to be equal to e f, e being the mean planet and f the true on the first day $e\hat{E}_1 f = P_1\hat{E}_2 d$. But E_1 e 11 E_2 P_1 $\therefore$ $e E_1 f = E_1\hat{P}_1E_2$ $\therefore$ $P_1\hat{E}_2 d = E_1\hat{P}_1E_2$ and alternate angles being equal E_1 P_1 11 E_2d. P_1 a is the H sine of P_1 d where P_2 b is the H sine P_2 d. Looking upon P_1P_2 as an increment in P_1 d i.e. looking upon the Kendragati $P_1\hat{E}_2P_2$ as an increment in S'ighraphala, Bhāskara uses the method of Bhogyakhanda sphuti Karana to obtain the Sphutakendragati. From the figure.

$$P_2c = P_2b - P_1a = H\sin(G + \delta m) - H\sin E$$

$$= \frac{H\sin E\ H\cos\delta m + H\cos E\ H\sin\delta m}{R} - H\sin E$$

Taking H cos δ M = R and H sin δm =

$P_2c = \frac{H \cos E \delta m}{R}$ This is at the end of $E_1 P_2$ i.e. at the end of K; so to get the corresponding chord in the deferent we do Karnānupāta so that the result is $\frac{H \cos E \delta m}{R} \times \frac{R}{K} = \frac{H \cos E \delta m}{K}$ as given by Bhāskara. We have cut short Bhāskara's method of Bhōgyakhanda Sphutīkaraṇa to make it clear to a modern student. Since P_2c is small, passing on from the H sine to the arc is not necessary, for, the H sine of a small arc is equal to the arc itself.

Bhāskara's argument, however, is as follows:—"If for 225, we have Bhōgya Khanda, what for δm?" The result is $\frac{B \times \delta m}{225}$; Then B is rectified as follows:—" When the H cosine E is equal to the radius, i.e. initially in the H sine table, the Bhōgyakhanda is 225, then what is it for H cos E?" The result is $\frac{H \cos E \times 225}{R}$ which we have to substitute for B. Then if this be at the end of K what is it at the end of R?" The result is $\frac{H \cos E \times 225}{R} \times \frac{\delta m}{225} \times R = H \cos E \times \delta m$

Before we proceed to explain "
we shall explain what Bhāskara pointed out as a mistake in Lallācharya.

Verse 40. Let Mathematicians understand that what formula was given by Lallācharya for S'ighragatiphala is not correct. When the anomaly is 90° or 270°, the gatiphala vanishes and there will be gatiphala at the points where it ought to be Zero according to his formula.

Comm. Ref. verse 45, Spaṣṭādhikāra, Siṣya Dhīvṛddhida

i.e.

$$\frac{\quad) \times R}{K} \times \frac{\text{S'īghraphala Bhōgyakanda}}{225}$$

= S'īghragatiphala.

Here the quantity within the brackets is δm. What Lallācharya had in his mind is as follows. 'If for the Ādya Khanda 225 we have H cos E = R, what shall we have for the Bhōgya Khanda of the S'īghraphala ?" The result would be $\left[H \sin (E + 225) - H \sin E \right] \times \frac{R}{225} =$

$$\left(\frac{H \sin E \; H \cos 225 + H \cos E \; H \sin 225 - H \sin E}{R} \right) \times \frac{R}{225}$$

Taking H cos 225 = R and H sin 225 = 225 it would be $\frac{H \cos E \times 225}{} \times \frac{R}{225} = H \cos E$. So Lallācharya's formula would become $\frac{\delta m}{K} \times H \cos E$ as given by Bhāskara. The charge levelled at Lallācharya is due to the fact that by the word 'Ās'u-chāpa' by which Lallācharya meant

Bhāskara meant 'आशुकेन्द्रचाप'. When the Kendra = 90 or 270, the Bhōgyakhanda being Zero, the S'īghragatiphala would be Zero. Also where it ought to be zero namely $(90 + H \sin^{1} r)$ & $(270 - H \sin^{1} r)$ it would not be zero. If Lallācharya had really meant what Bhāskara allributed to him, the formula would be $\frac{H \cos m \delta m}{K}$

it is very unlikely that Lallācharya would have meant this wrong formula, for, even if K were taken by him to be steady, $\delta \left(\frac{r \, H \sin m}{K} \right) = \frac{r \, H \cos m \delta m}{K}$ and Lallācharya's formula does not contain 'r'. The only non-rigorous part in Lallācharya's formula is at the point where he took

H cos 225 = R and H sin 225 = 225 which is rather crude. Bhāskara of course improved on this crudeness by taking to be an increment in E.

In the commentary under this verse, Bhāskara, having misinterpreted Lallācharya's phrase आशुचाप as meaning आशुकेन्द्रचाप and not as आशुफलचाप which was in the mind of Lallācharya, goes on recounting examples where the wrong formula attributed to him would give wrong results. So, we need not enter into those details.

Now we shall correlate Bhāskara's formula with its modern counterpart. Assuming coplanar heliocentric circular orbits for planets, let us see at what points two planets appear mutually stationary, that is, have a Zero relative angular velocity before they appear mutually retrograde.

Let S = Sun, E = Earth, J = Jupiter, u = Earth's linear velocity, v = Jupiter's linear velocity r and R the orbital radii of the Earth and Jupiter respectively. Let EE^1 and JJ^1 be perpendiculars to EJ so that when the relative velocity of Jupiter with respect to the Earth perpendicular to EJ is Zero, Jupiter will appear stationary as seen from the Earth. This

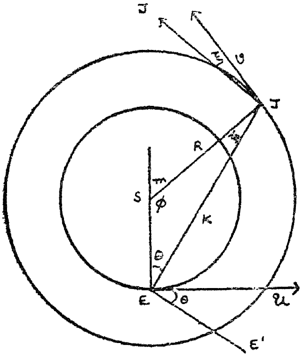


Fig. 17

means that $u \cos \theta + v \cos \xi = 0$ I

$$\therefore \quad \frac{u}{v} = \frac{-\cos\xi}{\cos \theta} \qquad \text{II}$$

But from triangle ESJ,

$$R \cos \phi + K \cos \theta = r \qquad \text{III and}$$

$$r \cos \phi + K \cos \xi = R \qquad \text{IV}$$

From III & IV $\frac{\cos \xi}{\cos \theta} = \frac{r \cos \phi - R}{R \cos \phi - r}$ V

Equating $\frac{\cos \xi}{\cos \theta}$ from II and V

$$-\frac{u}{v} = \frac{r \cos \phi - R}{R \cos \phi - r} \text{ so that } \frac{ru + Rv}{rv + Ru} = \cos \phi$$

If m be the S'īghra anomaly $m = 180 - \phi$

$$\text{so that } \cos m = -\left(\frac{ru + Rv}{rv + Ru}\right) \qquad \text{VI}$$

Now we shall show that Bhāskara's formula accords with this Spaṣṭagati = S'īghragati $- \frac{H \cos E\, \delta m}{K}$. As per Bhāskara

$$\text{Spaṣṭagati} = 0 \text{ if S'īghragati} = \frac{H \cos E\, \delta m}{K} \qquad \text{VII}$$

ie. Jupiter appears stationary as seen from Earth, if S'īghragati $= \frac{H \cos E\, \delta m}{K}$

The angular velocities of Earth and Jupiter are respectively $\frac{u}{r}$ and $\frac{v}{R}$ so that the Sun's apparent velocity is also $\frac{u}{r}$ and δm = Kēndra gati = Sun's apparent velocity minus Jupiter's heliocentric velocity $= \frac{u}{r} - \frac{v}{R}$

∴ Substituting in VII

$$\text{S'īghragati} = \frac{u}{r} = \frac{H \cos E}{K}\left(\frac{u}{r} - \frac{v}{R}\right)$$

$$\therefore \frac{u}{r}\left(\frac{H \cos E}{K} - 1\right) = \frac{H \cos E}{K} \times \frac{v}{R}$$

$$\text{ie. } \frac{u}{r}\left(\frac{H \cos E - K}{K}\right) = \frac{H \cos E}{K} \times \frac{v}{R}$$

But H cos E = R cos E = R cos ξ (here) ξ standing for E and R cos ξ − K = − r cos θ

so that $\frac{u}{r} \times - r \cos \theta = u \cos \xi$

$$\therefore \quad u \cos \theta + u \cos \xi = 0$$

This is equation I derived from modern methods, so that Bhāskara's formula accords with I and the rest follows as before i.e. $\cos m = -\left(\frac{ru + Rv}{rv + Ru}\right)$

Substituting the values of r and R for each of the planets and noting that r = epicyclic radius or Antyaphalajyā R = 3438′, u = mean velocity of the Sun and v = mean velocity of the planet, we have the respective values of m when planets appear stationary or, what is the same we could more easily use equation I noting that $\theta = m - \xi$ and thereby get the values of m for stationary values.

We shall give here some points of observation pertaining to S'ighraphala and S'ighragatiphala.

We have $M_2 + E_2 = S$ I where M_2 = Mandasphutagraha and E_2 = the S'ighraphala where from we have $\delta M_2 + \delta E_2 = \delta S$ II i.e. Mandasphutagati + S'ighragati phala = Spastagati (a) Let E_2 be maximum so that $\delta E_2 = 0$, then $\delta M_2 = \delta S$. This means that in the heliocentric figures 11 and 12, at the points a and b, the S'ighraphala being maximum, the Mandasphuta gati will be itself the Spashtagati. This line ab, it will be seen corresponds with the so called "कक्षामध्यगतिर्यग्रेखाप्रतिवृत्तसम्पातरेखा" i.e. the line cutting the eccentric, drawn through the centre of the deferent.

(b) The planet begins to retrograde only after the Spashtagati vanishes i.e. after $\delta M_2 + \delta E_2$ becomes Zero. Taking δM_2 to be almost a constant since the Mandagati phala is small, the negative value of δE_2 must cancel δM_2 in order that the Spashtagati may be zero. δE_2 becomes

negative when the planet courses the arc of the smaller segment ab, because E_2 decreases from a maximum value to zero. As a matter of fact δE_2 negatively increases along ac and again increases from a negative minum at c to Zero at b as we course along cb (vide fig. 11 & 12). Thus the planet will assume zero velocity at two points symmetrical about c and in between ab. In other words the planet will be retrograde along the arc d_1 cd_2 not entirely along ab as some have misconstrued. Regarding E_2, it is clear that it is zero at S'. then gradually increases to a maximum as the planet traces S'a, the maximum being assumed at a, then it decreases from the maximum to zero as the planet courses the arc ac. and then increases from zero to a maximum at b and finally decreases from that maximum to zero at S' again. Keeping the Earth constant it will be noted that an Inferior planet always goes anticlockwise whereas a Superior planet always goes clockwise. Also it will be seen that the S'īghraphala is positive as the planet courses the arc S'ac, whereas it is negative as it courses cbS'.

(c) The values of the spashtagati at S' and C will be respectively putting H cos E = R in the formula.

$$\text{Spashtagati} = \text{S'īghragati} - \frac{H \cos E \delta m}{K}.$$ Here for S'īghragati we may put U and U − V for δm, and putting K = R + r at S' and R − r at C, the values of the spashtagati would be

$\dfrac{U - R(U - V)}{R + r}$ and $\dfrac{U - R(U - V)}{R - r}$ ie. $\dfrac{RV + ru}{R + r}$, $\dfrac{RV - rU}{R - r}$ respectively.

$$\frac{rU}{r} > \frac{RV - rU}{R - r} \text{ if } R^2V - rRV + RrU - r^2U >$$
$$R^2V - RrU + rRV - r^2U$$

ie. if rR (U — V) > Rr (V — U). U — V is + ve and equal to V — U.

So the positive velocity at S′ of the planet will be equal to its negative or retrograde velocity at C. A quantity assuming values K, O, – K,O,K must have a value numerically less than K in between. Thus the velocity direct or retrograde at any point of the orbit is less than the numerical value of the velocity at S′ and C.

Verse 41. Retrograde motion.

The planets Mars, Mercury, Jupiter, Venus and Saturn will be retrograde when the S′ighra anomaly assumes values 163, 145, 125, 165 and 113 respectively and the direct motion again ensues at (360—163), (360 —145), (360—125), 360 — (165) and (360 — 113) respectively.

Comm. Since we have had the formula VI

$$\cos m = -\left(\frac{rU + Rv}{rV + RU}\right) \text{ for stationary points,}$$

noting that cos (180 — θ) = — cos θ = cos (180 + θ) the stationary points are symmetrically situated with respect to S′ of figures 11 and 12. As mentioned before substituting the values of U, V, r, R for all the planets we can prove the veracity of Bhāskara's statement.

Verse 42. Heliacal rising and setting of planets.

Mars rises heliacally in the East by 28°, Jupiter by 14°, Saturn by 17° of S′ighra anomaly and set heliacally in the west by degrees which are the differences of the above and 360° respectively.

Comm. The Sun's velocity being greater than that of the Superior planets, the Sun overtakes them so that they set in west and rise in the East. When these planets are

situated within particular limits from the Sun, they will be invisible in the rays of the Sun. As these superior planets will be near the Sun, near the moment of conjunction, they will not be seen at conjunction and within particular limits from the position of the Sun. The limits cited above mostly depend upon their respective brilliance and to some extent upon their distances from the Sun too. Their brilliance again depends upon the extent of their gibbosity. The formula given for the phase in modern astronomy is phase $= \frac{1+\cos EpS}{2}$ and noting that $\widehat{Eps} = E_2 =$ the S'ighraphala, phase $= \frac{1+\cos E_2}{2}$. Hence the Superior planets are always gibbous ie. the disc illuminated will be always greater than $\frac{1}{2}$. Though at conjunction $E_2 = 0$ and the entire discs of the major planets will be illuminated, we cannot see them as they are immersed in the rays of the Sun. As they emerge out of conjunction gradually E_2 will be increasing so that the discs will be illuminated lesser and lesser gradually. But K decreasing, the brilliance will not be so much effected. Along the arc acb (figs. 11, 12) the planets gradually gain in illumination and will be brightest when they are in opposition both as cos E increases and K decreases. In other words the Superior planets appear more and more brilliant when they are retrograding, being most brilliant at 'C'. The spherical radii of Jupiter Saturn and Mars being in decreasing order, their brilliance will be in decreasing order so that they will be rising at distances from the Sun which are in increasing order. The inverse square law of courses works here but combining the two factors (1) the spherical radius of the planet and (2) the inverse square law, we may take it that Bhāskara's numbers cited above namely 28°, 17° and 14° for Mars, Saturn and Jupiter respectively accord with truth, for, these numbers are given by Bhāskara or his authority Brahmagupta only after observing the planets rising heliacally.

That the positions of the planets while setting or rising heliacally will be situated symmetrically with respect to the Sun, goes without saying.

But one thing. In the chapter called Udayāstāmayādhyāya, the degrees known as Kālāmsas which are given as the arcs between the planets and the Sun for heliacal rising or setting, are different from the numbers given above, for, the latter are the values of the S'ighra anomaly. In the case of Mars when the S'ighra anomaly is 28°, the S'ighraphala will be 11°, so that the apparent distance of the planet will be 28°−11°=17° which are the Kālamsas for Mars. Similarly in the case of Jupiter, when the S'ighra anomaly is 14°, the S'ighraphala will be 3°, so that the distance between the planet and the Sun as seen from the earth will be 11°, which are given as Jupiter's Kālamsas. Also in the case of Saturn, the S'ighraphala for 17° of anomaly will be 2°, so that the Kālamsas would be 15°.

Verse 43. Mercury and Venus rise in the West by 50° and 24° of S'ighra anomaly respectively, and set in the West by 155°, and 177° respectively. They rise in the East by 205° and 183° of S'ighra anomaly and set there by 310° and 336° respectively.

Comm. Regarding the Inferior planets, they rise heliacally in the East after Inferior conjunction and then they are retrograde. They attain gradually the maximum elongation in the East and after they revert to direct motion, their elongation gradually decreases. They then set in the East and heliacally rise thereafter in the West. There again their elongation attains a maximum value; then they begin to retrograde and gradually set in the West only to rise in the East. This is all clear from the heliocentric figure 11. When the S'ighra anomalies of Mercury and Venus happen to be respectively 50° and 24°, their S'ighraphalas would be 13° and 11° respectively, so that they are themselves the Kālamsas; in as much as in

the case of the Inferior planets, the S'īghraphala will be itself their elongation eastern or western. Then they rise in the West, being near Superior conjunction. When again their S'īghra anomalies equal respectively 155° and 177°, the same S'īghraphalas will arise so that they set heliacally in the West. Then as the S'īghra anomalies attain the symmetrical values on the other side ie. (360–155) and (360–177) ie. 205° and 183° the S'īghraphalas being the same, they rise in the East. Again when they attain the values (360–50) and (360–24) ie. 310° and 336°, they set in the East on account of the same S'īghraphalas or Kālamsas.

Verse 44. When the S'īghra anomalies have particular values, to decide when the planets rise or set heliacally, we have to take the difference of those particular values and the numbers given above for the respective S'īghra anomalies, convert them into minutes of arc and divide the results by the daily motion in the S'īghra anomalies in minutes of arc. Then we have the number of days in which the rising or setting takes place thereafter.

Comm. Suppose it is required when Mercury rises in the West. Suppose we want to compute this on a particular day when Mercury's S'īghra anomaly is x°. Then because we know that Mercury rises in the West when his S'īghra anomaly is 50, we have to calculate by how many days the difference $|x-50|$ of the S'īghra anomaly would be covered. Let the daily motion of the S'īghra anomaly be y′ per day, Then by rule of three $\frac{|x-50| \times 60}{y}$ will be the number of days before or after as the case may be for Mercury to rise in the West.

Verse 45. To obtain the Mean planet knowing the True.

Assume the True planet to be the Mean; compute the Manda and S'īghraphalas and applying them inversely,

we have an approximation of the Mean planets. Treating these as the Mean planets, again obtaining the Manda and Śīghraphalas and again applying them inversely and repeating the process till constant values are obtained, we have by this method of successive approximation the Mean planets required.

Comm. The method of successive approximation is clear.

Verse 46. To obtain the equinoctial shadow.

Convert the Ayanāmsas into minutes of arc, and divide by the mean daily motion of the Sun; then we have the number of days before the Meṣa or Tulā Samkrānti day, or before Makara and Karkataka Samkrānts days, when the Sun will be in equinoxes or Solstices respectively. The mid-day shadow of the Sun cast by the gnomon on such an equinoctial day, will give us the equinoctial shadow required.

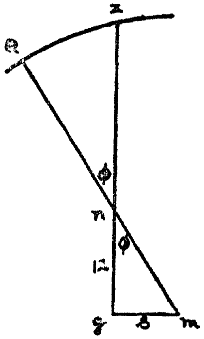


Fig. 18

Comm. The palabhā or equinoctial shadow as it is called is the length of the shadow cast by a gnomon taken to be of 12 units in length, (measuring the shadow also in the same units), at noon of an equinoctial day. In other words, if this shadow be of s units, clearly $\frac{s}{12} = \tan \phi$ (Vide fig. 18). The Ayanāmśas are the degrees of the arc of the ecliptic in between the Hindu Zero point of the Zodiac and the first point of Aries which is now behind the former due to the phenomenon known as the precession of the equinoxes. They are called Ayanāmśas because the solstices are also behind the Makara and Karkataka Samkrānti points of the Hindu Zodiac or points which have Hindu longitudes 270° and 90° resply, by the same arc. The Hindu astro-

nomers came to know that the solstices are preceding by observations made with the gnomonic shadow at mid-day around the solstitial days. The day on which the maximum mid-day shadow is cast by the gnomon is the Winter solstitial day whereas the day on which the mid-day gnomonic shadow is maximum in the southern direction (assuming the place to be of northern latitude and $\phi < \omega$) or minimum in the northern direction ($\phi > \omega$) is the summer solstitial day. Calculating the Sun's longitude on that day at noon, we know how far the solstitial points have preceded behind the points of the Hindu Zodiac which have Hindu longitudes 90° and 270°. The word **अयनांशाः** has therefore the meaning **अयनविलोमगत्यंशाः** where the Samāsa may be viewed as a

Verses 47, 48. Calculating the five fundamental H sines of a point of the Zodiac pertaining to a point of the ecliptic.

The declination has to be computed from the sum of the Hindu longitude of that point and the Ayanamsas. Similarly if it be required to find the time before or after the rise of a point of the ecliptic, we have to compute them from the sum of the longitude of that point and the Ayanamsas.

$$\frac{H\sin 24^\circ \times H\sin\lambda}{R} = I$$

$$H\sin^2\delta = H\cos\delta = \text{Dyujyā} \quad \text{II}$$

$$\delta = H\sin^{-1}(H\sin\delta) \quad \text{III}$$

$$\frac{s}{12} \times H\sin\delta = \text{Kujyā} \quad \text{IV}$$

$$\times R$$

$$H\cos\delta$$

$$H\sin^{-1}(\text{Charajyā}) = \text{Charam} \quad \text{VI}$$

Comm. (1) Let rA⊙ be the ecliptic where r = Vernal Equinox, A = first point of the Hindu Zodiac,

in between r and ⊙, not shown in the figure ⊙ = the Sun. Let rA = a° = Ayanamsas defined before so that r⊙ =(a+λ). From the spherical triangle r⊙M, by Napier's rule, $\sin \delta = (\sin \lambda + a) \sin \omega$ which in Hindu trigonometry becomes $H \sin \delta =$

$$\frac{H \sin (\lambda + a) \times \sin \omega}{R}$$

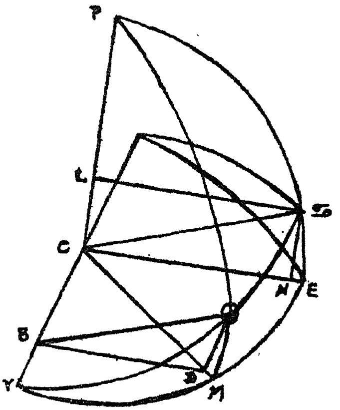


Fig. 19

In Hindu Astronomy the obliquity of the ecliptic was taken to be 24°. The value of the obliquity is now 23°–27′ approximately and it has been know that this has been decreasing. At the time when the Hindu Astronomers observed this, it should have been greater than 23°–27′ so that if it was taken to be 24°, their observations were not far from truth. This means that the antiquity of Hindu Astronomy might be far more than what the Moderns estimate it to be.

(2) How the formula I was derived in Hindu trigonometry was as follows. Let ♋ be the summer solstice ♋C, ⊙B, the perpendiculars dropped from ♋ and ⊙ on the line of intersection of the Equatorial and Ecliptic planes namely rBC. Let perpendiculars be dropped from ♋ and ⊙ on the plane of the Equator. Let them be ♋N, ⊙D, Join NC and DB. Then $\widehat{\odot BD} = \widehat{♋CN}$ = the dihedral angle between the two planes = ω; C♋ = R since in Hindu trigonometry H sin 90° = R. Also ♋N = H sin ♋E (in Hindu trigonometry) = H sin ω; ⊙D = H sin ⊙M = H sin δ; ⊙B = H sin r⊙ = H sin (λ + a) where λ is the Hindu longitude of the Sun and a = Ayanamśas. It will be seen that ⊙D is a segment of the

line of intersection of the planes PCM a plane perpendicular to the plane of the Equator, and ⊙BD a plane perpendicular to the Ecliptic plane. Since ♋C $\parallel$ ⊙B and ♋N $\parallel$ ⊙D and BD⊙ = CN♋ = 90°, so the two triangles are congruent

$$\therefore \frac{\odot B}{♋C} = \frac{\odot D}{♋N} \text{ ie. } H\sin(\lambda + a) = \frac{R \times H\sin\delta}{H\sin\omega}$$

$$\therefore H\sin\delta = \frac{H\sin(\lambda + a)\, H\sin\omega}{R}$$

$$NC = ♋L = \sqrt{C♋^2 - ♋N^2} = \sqrt{R^2 - H\sin^2\delta} =$$

H cos δ. ♋L is called Dyujyā as explained below in note (3).

Another way of looking at the similarity of the triangles C♋N and B⊙D is from the fact that they are formed by the intersection of parallel planes, both of which are perpendicular to the plane of the Equator. This idea will be elaborated when we explain fig. 21, wherein the so-called latitudinal triangles will be shown to be formed by the intersection of the plane of the Equator and planes of diurnal circles with the planes of the horizon and the prime vertical. In fact, the plane of a great circle and the parallel planes of the corresponding small circles form the same dihedral angle with the planes of the celestial sphere namely the planes of the meridian, horizon and prime-vertical so that right-angled triangles formed by their intersection will be all similar.

Formula II is derived from the formula H $\sin^2\delta$ + H $\cos^2\delta = R^2$ derived from fig. 6 from which formula, formula III follows.

(3) Why H cos δ is called Dyujyā in formula II is clear from fig. 20 wherein p is the celestial pole, (C) is the

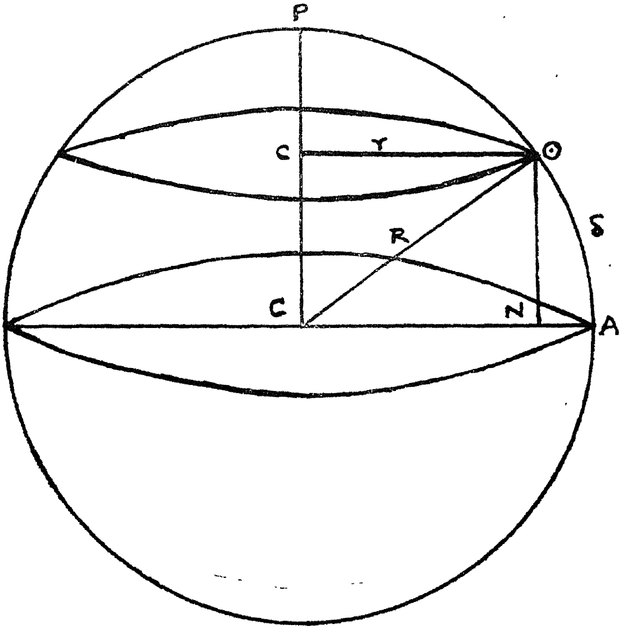


Fig. 20

celestial Equator and (C) is the diurnal circle of a celestial body say the ⊙ ie. the Sun. Let ⊙N be the perpendicular dropped from ⊙ on the plane of the Equator so that ⊙N = H sin ⊙A = H sin δ CN = H cos δ = C⊙ = radius of the diurnal circle called Dyujyā (Dyu = day) द्युवृत्तत्रिज्या = द्युज्या (Madhyamapadalōpi Samāsa) formulae IV, V and VI will be dealt with in the next chapter Tripras'nādhyāya more elaborately but one has to understand what Charajyā is to follow the subsequent verses of this chapter so that we shall give its location and definition in the light of fig. 21. Let fig. 21 represent the celestial sphere in which SEN = Horizon, Z = Zenith, N = Nadir Ez = prime vertical EQR = celestial equator,

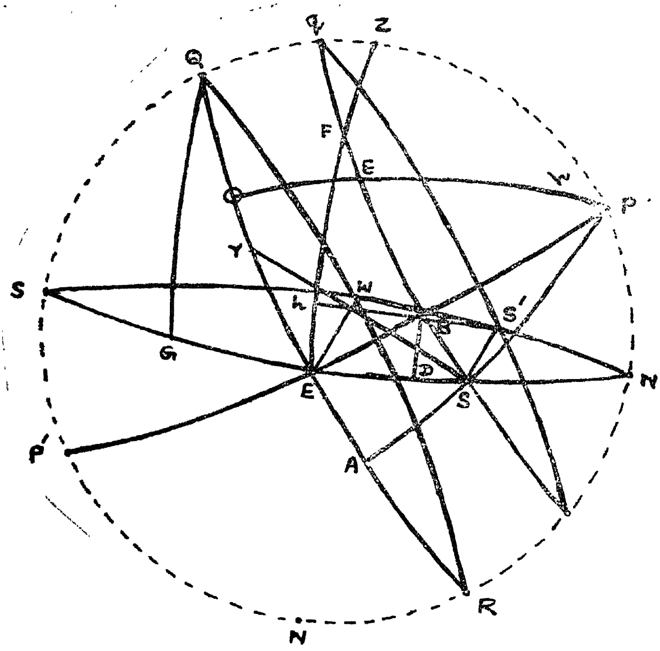


Fig. 21

PP′ = Polar axis, SBS′ = the diurnal circle of a celestial body S; EW = the East-West line, SS′ = Udayāstasūtra or the join of the rising and setting points S, S′. This SS′ is evidently a diameter of the diurnal circle which is bisected by the plane of the horizon; PEP′ is called the unmandala or the Equatorial horizon. PSA is the declination circle of S cutting the Equator in A. EA is called the charam whose H sine is called Charajyā. The H sine of SB in the diurnal circle is called Kujyā so that as corresponding lines in the diurnal circle and the Equator stand in the ratio H cos δ : R,

$$\frac{\text{Kujya}}{\text{Charajyā}} \quad \frac{\text{H cos}}{\text{R}} \qquad \text{I}$$

In the beginning of the Tripras'nādhyāya, Bhāskara says "साक्षे देशे खगोलवलयानां, तिरश्चीनभगोलवलयानां च संपातात्त्र्यस्राणि क्षेत्राण्युत्पद्यन्ते, तान्यक्षक्षेत्रसंज्ञानि ie. In a place having a latitude the diurnal paths of stars and planets will be inclined to the fundamental circles of the celestial sphere namely horizon, meridian and prime vertical and so their intersection gives rise to what are called latitudinal triangles, in which the angles would be ϕ, 90-ϕ and 90 where ϕ is the latitude. Thus in fig. 21 the projections of the triangles ESB, EDB, DSB, EDF, EBF, ESF and EQG on the meridian plane will be all triangles in which the angles will be ϕ, 90-ϕ and 90° so that they are all latitudinal triangles. These are all similar to the fundamental gnomonic triangle gmn of fig. 18 wherein also the angles are ϕ, 90-ϕ and 90°. Here, there is one important point to be observed. We have said "their projections are all similar". In fact the corresponding spherical triangles enumerated above are all apparently similar, though they are not viewed in modern astronomy as regular spherical triangles, because all the three sides of the triangles are not arcs of great circles. It is not possible to apply Napier's rules to these triangles for the reason mentioned above. None the less, the property of similarity of the projected right angled triangles is made use of in Hindu Astronomy to obtain the magnitudes of the sides of the projected triangles.

EW is called the prāk-pratichī sūtra, SS′ the Udayāsta sūtra; similarly if lines through F, B, D, A parallel to EW be drawn, these sūtras will intersect the meridian planes in points which constitute the projected triangles mentioned above. Let us study these triangles which we connote by the same letters with lowered indices. Thus for example $E_1 S_1 B_1$ is the projected triangle of ESB where of course E_1 will be the centre of the celestial sphere. The lines drawn parallel to SS′ or EW through B, D etc. will be denoted as BB′, DD′, FF′ etc.

In the triangle $E_1 S_1 B_1$, $E_1 S_1$ is called Agrajyā, $S_1 B_1$ Kujyā, and $E_1 B_1$ Krāntijyā. Of the sides of ESB. ES and EB are arcs of great circles whereas SB is the arc of a small circle. In the projected triangle $E_1 S_1 B_1$, $E_1 S_1$ will be equal to the perpendicular from S on $E\omega$, which will be H sin (ES) and is called Agrajyā; $E_1 B_1$ will be equal to the perpendicular from B on $E\omega$ which will be H sin BE and as such called Krāntijyā. But $S_1 B_1$ which is equal to the perpendicular from S_1 on BB′ the diameter of the diurnal circle is the H sine of SB in the diurnal circle and is called Kujyā. It will be noted that the perpendiculars from S on $E\omega$, and BB′ and the perpendicular from B on $E\omega$ do not form a triangle by themselves but by the theorem of three perpendiculars, if SL be the perpendicular from S on the plane of the unmandala ie. the great circle EBP, and if LM be perpendicular from L on $E\omega$, SM will be perpendicular on $E\omega$. Here the perpendicular SL will be the Kujyā, and SM the Agrajyā, whereas LM is not actually the H sine of BE but is equal and parallel to it.

Similarly take the spherical triangle SAE. This is a regular spherical triangle because the three sides are arcs of great circles. Napier's rules can be applied to this triangle and we have the formula sin SA = sin SE sin $S\hat{E}A$ or in Hindu form $H \sin SA = \dfrac{H \sin SE \times H \sin \hat{E}}{R}$

or $RH \sin \delta = \text{Agrajya} \times H \cos \phi$ ie. Agrajyā =

$$\frac{RH \sin \delta}{H \cos \phi} \qquad \text{II}$$

Again $\sin EA = \underline{\tan SA} \times \tan \hat{E}$ where H sin EA is called Charajyā and $\tan E = \cot \phi$ so that Charajyā = $\underline{\tan \delta}\ \tan \phi$ in modern form, and the Hindu form is $R \tan \delta \tan \phi$

From I $\frac{\text{H cos } \delta}{\text{R}} = \frac{\text{Kujyā}}{\text{Charajyā}}$ so that

$$\text{Kujyā} = \frac{\text{H cos } \delta}{\text{R}} \times \text{R } \underline{\tan \delta} \tan \phi = \underline{\text{H sin } \delta} \tan \phi \quad \text{III}$$

In Tripras'nādhyāya, the elements of all the eight latitudinal triangles are found by using their similarity with the fundamental gnomonic triangle. The important elements that will enter into computation are (a) Charajyā (b) Kujyā (c) Agrajyā (d) Taddhriti ie. $S_1 F_1$ the projection of SF on the plane of the meridian (e) Sama-S'anku $= E_1 F_1 =$ H sin EF (f) Krāntijyā $= E_1 B_1 =$ H sin EB $=$ H sin δ (g) Lambajyā $=$ H sin QS $=$ H cos ZQ $=$ H cos ϕ (h) Akshajya $=$ H sin ZQ $=$ H sin ϕ (i) $B_1 D_1$ $=$ Ud-Vritha-S'anku $=$ H cos ZB (j) Dinārdha-S'anku $=$ H cos Zq. The following points will be noted.

(i) In the triangle $E_1 Q G_1$, the projected triangle of EQG on the meridian plane, noting that E_1 is the centre of the celestial sphere $E_1 Q = R$, $QG_1 =$ H cos ϕ and $E_1 G_1 =$ H sin ZQ $=$ H sinϕ.

(ii) $S_1 f_1 =$ H sin SB $+$ H sin fB (both the H sines pertaining to the diurnal circle.

(iii) Sama S'anku is the H cosine of the Zenith-distance when the Sun or celestial body is on the prime-vertical.

(iv) BD is an arc of the great circle ZB, so that the Unmandala S'anku is the H cosine of ZB.

(v) Dinārdha — S'anku $=$ H cosine ZQ.

(vi) These S'ankus are the H sines of altitudes or H cosines of Zenith-distances and they are in the planes of the respective great circles.

(vii) Lambāmsa-chāpa is the arc of the colatitude QS where S is the South point so that Lambajyā is the H sine of QS or H cosine of ZQ. This Lambajyā will be seen to be the diameter of a small circle parallel to the Equator and passing through Z.

Verses 49–51. A different method of obtaining chara.

This chara can be had by the so-called chara-segments of the locality using a process similar to that of finding the H sines of the smaller table of nine H sines using $\frac{\lambda}{3}$ where λ is the Sāyana longitude of the Sun (i.e. The Hindu longitude plus the arc of ayanāmśas is called the Sāyana longitude or the modern longitude measured from ♈ along the ecliptic to the Sun). Find the charas of 30°, 60° and 90° of the ecliptic measured from ♈. Subtract the first from the second, the second from the third. Thus we have C30°, C60°–C30°, C90°–C60°, (where Cθ° signifies the chara of θ°) which are called chara-Khandas or chara-segments. The equinoctial shadow multiplied by 10, 8, $3\frac{1}{3}$ gives the approximate values of the chara-segments in Vinādis (a sidereal day is divided into 60 nādis and each nādi consists of 60 Vinādis). The chara-segments thus measured in Vinādis are rather approximate. If further exactitude is required, better take the arc in units each of which rises in $\frac{1}{6}$th of a Vinādi (This $\frac{1}{6}$th part of a Vinādi is known as a prāna i.e. the duration of the interval between two inhales of a healthy person reckoned as 4″ of time).

Comm. The charas of 30°, 60° and 90° of modern longitude are the values of the arc EA, (Ref. Fig. 21) when *S* has longitudes 30°, 60° and 90°. We have the folmula

$$\text{Charajyā} = R \tan \delta \tan \phi$$

Taking the equinoctial shadow equal to 1 Angula means $\tan \phi = \frac{1}{12}$. As charajyā is proportional to $\tan \phi$, for any equinoctial shadow of s angulas, $\tan \phi$ being equal to $\frac{s}{12}$ the charajya got above is to be multiplied by *S* only to obtain the charajyā in any place where the equinoctial shadow is s angulas. Putting the modern longitudes equal to 30° & 60°, if the corresponding declinations be δ_1, δ_2 $\sin \delta_1 = \sin 30 \sin \omega$, $\sin \delta_2 = \sin 60 \sin \omega$. Taking $\omega = 24°$ and applying logarithmic tables

$\log \sin \delta_1 = 9.6990 + 9.6093 = 9.3083$ so that $\delta_1 = 11°\text{–}44'$

$\log \sin \delta_2 = 9.9375 + 9.6093 = 9.5468$ so that $\delta_2 = 20°\text{–}38'$

Now from the formula for charajya cited above viz. H sine (chara) = R $\tan \delta$ $\tan \phi$ or sine (chara) = $\tan \phi \tan \delta$, putting $\tan \phi = \frac{1}{12}$ and applying tables, using the values of δ got above,

charajya for 30° $= \frac{\tan 11°\text{–}44'}{12}$

and charajyā for 60° $= \frac{\tan 20' - 38'}{12}$ so that

log (sine chara) = 9.3175−1.0792 for 30° and for 60°

log (sine chara) = 9.5758−1.0792

∴ Chara for 30° or C (30)° = 59′ and C (60°) = 1° - 48′

Converting these arcs into their rising times at the rate of 6′ per Vinadi, we have C (30°) = 10, and C (60°) = 18

Noting $\delta_3 = \omega$, sin (chara) for 90° $= \frac{\tan 24°}{12}$ so that

log sin (C 90°) = 9.6486 − 1.0792 = 8.5694 so that

C (90°) = 2°–8′ = $21\frac{1}{3}$ Vinādis.

Thus (C $\overline{30}$) = 10, C (60)−C (30) = 18 − 10 = 8 C (90°) − C (60°) = $21\frac{1}{3} - 18 = 3\frac{1}{3}$ so that the chara Segments are respectively 10, 8, $3\frac{1}{2}$ as given by Bhāskara.

For a given place of equinoctial shadow equal to s″, we have to multiply 10, 8, $3\frac{1}{2}$, by s, which will be the chara-segments for the place.

The meaning of the first half of the verse 49 is as follows.—Let the equinoctial shadow for a place be 3 Angulas. Then the chara-segments for that place are 30, 24, 10. These are three in number for a longitude λ of 90°. If the longitude be 44° (say) then proceed as we have done to find sin 24°, using the method of Bhogya Khanda Sphutīkaraṇa with respect to the table of H sines namely 21, 20, 19, 17, 15, 12, 9, 5, 2. Proceeding as directed the Sphuta Bhogya Khanda for 14° is $\left(\frac{3 \times 14}{30} = \frac{7}{5}; \ 30 - \frac{7}{5} = 28\frac{3}{5}\right.$

$$\frac{14 \times 28\frac{3}{5}}{30} = \frac{2002}{150} = 13\frac{1}{3}; \quad 30 + 13\frac{1}{3} =$$

Proceeding according to the modern formula we have

Charajyā $= \tan \delta \tan \phi$ where $\phi = \frac{3}{12}$, and $\sin \delta =$ $\sin 44° \sin 24°$

$\log \sin \delta = 9.8418 + 9.6093 = 9.4511; \ \delta = 16°\text{–}25$

$\log \tan \delta = 9.4693 \quad \therefore \log \sin (\text{chara}) = 9.4693 + \log \tan \phi$

$= 9.4693 + \log \frac{3}{12} = 9.4693 - .6021 = 8.8672$

$\therefore$ Chara $= 4°\text{–}14' = \frac{254}{6} = 42\frac{1}{3}$ Vinadis whereas we have got by the Hindu method $43\frac{1}{3}$ which is near the truth.

Verse 52. To find the durations of day and night.

Fifteen ghatis increased or decreased by the Chara-nadis, according as the Sun is in the northern hemisphere or southern, gives half the day of the locality and the difference of 30 nādis and the above half-day gives half the duration of night.

Comm. Ref. fig. 21. Let S be the Sun rising in the hemisphere when his declination is north. Then

from the figure $\widehat{APQ}$ is the rising hour-angle $= \widehat{APE} + \widehat{EPQ} = \widehat{APE} + 90°$ where 90° correspond to 15 nādis. So we have to add $\widehat{APE}$ expressed in nādis equal to the rising time of AE. Let us find the duration of the day for the place where s = 3″ and when λ of s = 44°; the latitude of the place will be 14° (from tables) when λ 44°, we have found above that 42–20 Vinadis is the chara expressed in time. Hence half the day = 15–42–20 or duration of day 31–25; duration of night = 28–35.

Note (1) In modern astronomy we have the formula $\cos h = -\tan \phi \tan \delta$ where h is the rising hour angle. Putting $h = H + 90°$ where H stands for the arc EA of fig. 21 $\cos (90+H) = -\sin h = -\tan \phi \tan \delta$ so that $\sin H = \tan \phi \tan \delta$ ie. $H \sin (\text{Chara}) = R \tan \phi \tan \delta$ which accords with the formula found before. The word chara used for EA means etymologically रविसञ्चारवशेन दिन-विकारः i e. the variation in 15 ghatis of the eqiunoctial half-day on account of the Sun's variation in declination,

(2) If $\phi = 0$, Chara = 0 so that duration of half-day is 15 ghatis ie. on the terrestrial equator, whatever be the Sun's declination, the length of the day will be always 12 hours.

(3) Let $\delta = 0$ so that chara = 0 ie. whatever be the latitude (provided $\phi \not> 90-\delta$ as we shall see shortly). the day and night will be each of 12 hrs.

(4) Let δ be negative, so that the arc connoting chaaa ie. EA will be above the horizon, and consequently the duration of half-day will be less than 6 hrs. by the time that is taken for EA to rise. This can be seen otherwise also as $\cos h = -\tan \phi \tan \delta = +$ ve so that $h < 90°$ which means half-day is less than 6 hrs.

(5) We could also treat the case when ϕ is negative, but as this case is not in the purview of Hindu Astronomers who had only India in their mind and as such were concerned primarily with positive latitudes. If, however, we consider a negative latitude, when δ is + ve, cos h will be positive and if δ is − ve, cos b will be negative. This means that when the Sun is in northern latitudes, the southern latitudes will have their day less than 12 hours and when the Sun is in southern latitudes, their day will be greater than 12 hrs.

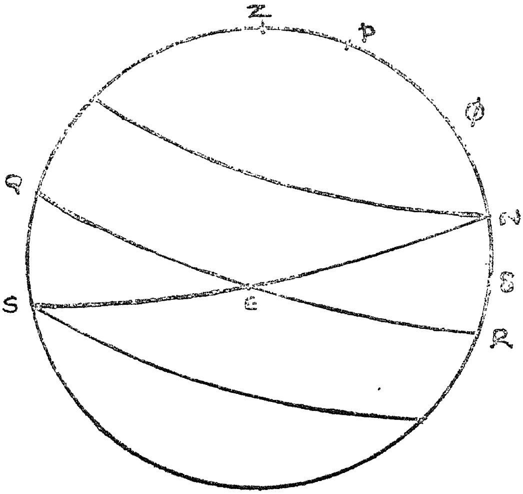


Fig. 22

(6) Let $\phi + \delta = 20°$. (Ref. fig. 22) ie. imagine the Sun to rise at N, the north point so that $RN + NP = \delta + \phi = 90°$. Then the Sun's diurnal path will be entirely above the horizon, which means that what is called 'perpetual day' begins for that place on that day and lasts as long as $\delta \geqslant 90 - \phi$. For the same place, let $\delta = -(90 - \phi) = QS$. In this case the Sun sets at S, and what is called perpetual night begins and lasts till the southern declination of the Sun is greater than $90 - \phi$. The duration of perpetual day can be found as follows which applies to the perpetual night as well.

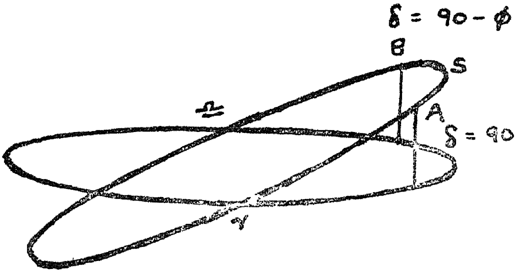


Fig. 23

(Ref. fig. 23). Let A be the point at which the declination is $90 - \phi$; let S be the summer solstice and let B be the point where again the declination is equal to $90 - \phi$. So long as the Sun traces the arc $AB = 2\ AS =$

$(90-\gamma A)$, there will be perpetual day. But $\sin \delta = \sin \lambda \sin \omega$ so that $\sin \lambda$

$$= \frac{\sin \delta}{\sin \omega}. \text{ Putting } \delta = 90 - \phi, \sin \lambda = \frac{\cos \phi}{\sin \omega};$$

$$\text{Sin } \lambda = \sin \gamma A = \cos AS \quad \therefore \cos AS = \frac{\cos \phi}{\sin \omega}$$

$\therefore 2\text{ AS} = 2 \cos^{-1}\left(\frac{\cos \phi}{\sin \omega}\right)$. Supposing AS expressed in degrees and assuming the Sun goes along the ecliptic with uniform motion, since he takes $365\frac{1}{4}$ days to trace 360°, to trace 2 AS, he takes $\frac{2\text{ AS}}{360} \times 365\frac{1}{4}$ days $= \frac{365\frac{1}{4}}{180} \times \cos^{-1}\left(\frac{\cos \phi}{\sin \omega}\right)$ which is the length of the perpetual day.

Verse 52. The correction known as Chara.

The daily motion of the planet being multiplied by the chara expressed in asus and divided by the asus in a day viz. 21659 and the result being subtracted from or added to the planetary position at Sunrise according as the Sun is in the northern or southern hemisphere. The result is to be added to or subtracted from the planetary position at Sunset.

Comm. The mean planets computed hold good at the Sun-rise at Lanka i.e. at zero latitude; they have to be converted to hold good at the local Sun-rise. In other words in fig. 21, the mean planet computed is B which is on the Lanka horizon, whereas we have to get S, the same planet on the local horizon. The position of S is earlier than that of B by the time interval indicated by the arc SB which is measured by EA the chara because the position B on the Lanka horizon is later than the position S on the local horizon. If the mean planet moves in a day of 21659 asus by the arc denoting its daily mean motion, by how much does it move in the time of chara expressed in asus? The result is

$$\frac{\text{Chara in Asus} \times \text{mean daily motion}}{21659}.$$

This result is to be subtracted from the position of B to get the position of S. If the position B′ which is the setting position at Lanka, and if we have to get S′ the local setting position, in as much S′ is later than B′ by the same arc, we have to add the above result to the position B′ to get S′. Here the asus in a day is given to be 21659 and not $60 \times 60 \times 6 = 21600$ because the mean daily motion is during the course of the Sāvana day i.e. the true solar day and not during the course of a sidereal day of 21600 asus. The Sāvana day exceeds the sidereal by the time taken by the arc moved by the Sun during that day which is very approximately 59′. Since a minute of arc of the Equator rises in an asu i.e. in 4″, so 59′ of the Sun's motion is covered in 59 asus which is to be added to 21600 asus of the sidereal day to get the Sāvana day approximately. It will be noted that this correction of chara in the planetary positions is due to latitude of the place and if the latitude is zero, it need not be done. Also if $\delta = 0$, it need not be done, for, then, the Sun or the celestial body whose $\delta = 0$ will be rising at E itself in which case the chara EA will be zero.

Verses 54, 55. The H sines of 30°, 60° and 90°, being squared and decreased by the squares of their respective declinations, the square-root of the differences being taken, and the result being multiplied by the radius, is to be divided by the respective H cosines of the declination. The first of the three results, the difference of the second and the first and the difference of the third and the second will give us the rising times of what are called the Sāyana rasis of Mesha, Vrishabha and Mithuna; their reverses will then give the rising times of the next three; then the original ones those of the next three and again then reverses those of the last three.

Comm. There are twelve Rasis in the Zodiac of equal interval. Measuring from *r*, and taking arcs of the

ecliptic successively each of 30°, we have what are called Sāyana Rasis or Rasis taking the Ayanāms'a into consideration or in other words measuring from r. On the other hand successive arcs each of 30° measured from the zero point of the Hindu Zodiac, constitute what are called the Nirayana Rās'is. The words Meṣa, Vriṣabha etc. signifying the shapes of the constellations apply strictly to the Nirayana Rās'is. But nonetheless, by the convention what is called Upachāra, we name the Sāyana Rās'is also by the same names. In as much as the zero-points of the Hindu Zodiacs is ahead of the modern zero-point by an arc which is the arc of Ayanamsa or accumulated precession, the words Sāyana and Nirayana came into vogue. Once upon a time approximately in Varāha's time or rather 499 A. D, the two zero-points coincided and then the Sāyana and Nirayana Rās'is were the same. Gradually on account of the phenomenon of precession r preceded, and today the distance between r and the Hindu zero-point is about 20°-30′. Of late, there has been a big controversy as to what exactly the arc is and the Calendar reform committee has adopted a value far more than what could be justified. The present author opines, that we have no right to set aside a statement of no less an astronomer than Varāhamihira who stated explicity 'आश्लेषार्धात् दक्षिणमुत्तरमयनं रवेर्धनिष्ठाद्यम्, नूनं कदाचिदासीत् येनोक्तंपूर्वशास्त्रेषु, साम्प्रतमयनं सवितुः कर्कटकाद्यं मृगादितश्चाऽन्यत् उक्ताभावो विकृतिः प्रत्यक्षपरीक्षणैर्व्यक्तिः" i.e. True it is that once upon a time, the Sun began his southern journey when he was mid-way the constellation of Asleṣa and his northern when he was at the beginning of Dhaniṣtha, because it was stated in ancient texts. (The allusion is to Vedānga Jyotiṣa where it was stated as such); but now the southern journey of the Sun begins from the point marking $\frac{3}{4}$ of the constellation of punarvasu which is the beginning point of the Karkata Rās'i and his norther from the beginning of Makara i.e. from the point marking $\frac{1}{4}$th of the constellation of Uttarāṣādha; there has been a change from what was

stated in the ancient texts ; let people verify this by actual observation ". Accepting an Ayanāms'a which goes against the statement of this great astronomer, who said that he observed and called upon others to observe, is really unwarranted, especially when the adopted Ayanāms'a of the Calendar reform committee goes on the basis of surmises and consensus. We shall deal with this topic in further detail in an appendix to this work, because it is a really important issue and has been wrongly solved. The importance of knowing the exact value of the arc is clear when we observe that from the correctly computed planetary positions of modern astronomy this ayanāms'a is being subtracted to give the positions measured from the Zero-point of the Hindu Zodiac. An error in the Ayanāms'a therefore vitiates the positions obtained by the above method.

Coming to the point, the problem on hand is to obtain the rising times of the Sāyana Rās'is at a place of zero latitude i.e. what are called Lankōdaya times of Sāyana Rās'is (Ref. fig. 21). Let rS = 30° so that rA the equatorial arc gives the rising time of rS. We could have the magnitude of this arc by the formula derived from Napier's rules namely $\cos \omega = \tan \alpha \tan \lambda$ I.
But as in Hindu trigonometry the tangent functions of angles are not used, Bhāskara gave the following formula

$\text{Sin}\ \alpha = \dfrac{\sqrt{\sin^2 \lambda - \sin^2 \delta}}{\cos \delta}$ II. This formula could be derived easily from formula I and the formulae $\cos \lambda = \cos \alpha \cos \delta$ III and $\sin \delta = \sin \lambda \sin \omega$ IV ; for multiplying the right-hand sides of I and III we have $\cos \omega \cos \lambda = \dfrac{\sin \alpha \cos \delta}{\tan \lambda}$ i.e. $\sin \lambda \cos \omega = \cos \delta \sin \alpha$

$\therefore \sin \alpha = \dfrac{\sin \lambda \cos \omega}{\cos \delta}$ V. But from IV $\sin \omega =$

$\frac{\sin \delta}{\sin \lambda}$ so that $\cos \omega = \sqrt{1 - \frac{\sin^2 \delta}{\sin^2 \lambda}} =$

$\frac{1}{\sin \lambda} \sqrt{\sin^2 \lambda - \sin^2 \delta}$. Substituting in V for $\cos \omega$

we have

$$H \sin \alpha = R \frac{\sqrt{H \sin^2 \lambda - H \sin^2 \delta}}{H \cos \delta} \sin \alpha =$$

$$\sin \lambda \times \frac{1}{\sin \lambda} \frac{\sqrt{\sin^2 \lambda - \sin^2 \delta}}{\cos \delta} \qquad \frac{- \sin^2}{\cos}$$

$H \cos \delta$ as stated.

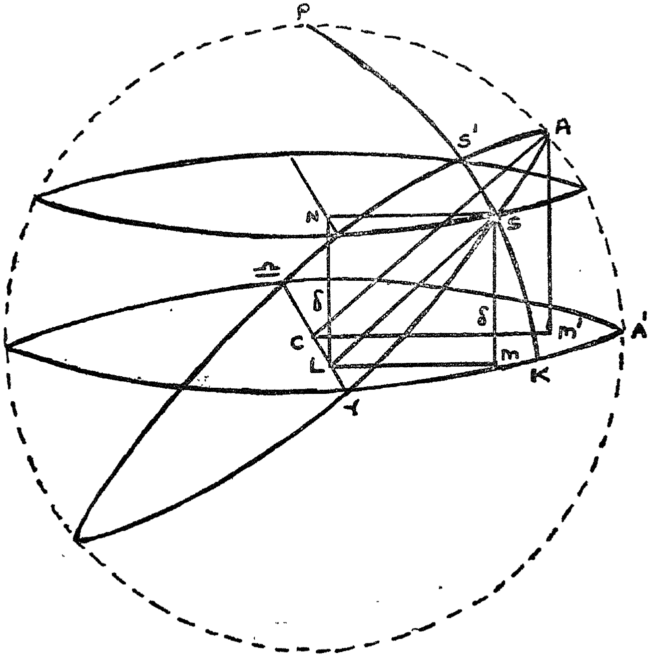


Fig. 24

But let us see how this formula was derived by the Hindu astronomers. (Ref. fig. 24). Let ♈S be an arc of the ecliptic equal to λ. Let SM be dropped perpendicular from S on the plane of the Equator so that SM = H sin δ.

From M drop a perpendicular ML on $r\simeq$ so that SL will be perpendicular from S on $r\simeq$ by the theorem of three perpendiculars. SL = H sin λ. Also ML will be equal to the perpendicular from S on the diameter of the diurnal circle of S parallel to $r\simeq$ so that ML = SN = H sine of the arc in the diurnal circle corresponding to Kr

$\therefore$ $SW^2 = SL^2 - LN^2 = H\sin^2\lambda - H\sin^2\delta$

$\therefore$ $SN = \sqrt{H\sin^2\lambda - H\sin^2\delta}$

$\therefore$ The length of the perpendicular from K on $r\simeq$ =

$R \times \frac{\sqrt{H\sin^2\lambda - H\sin^2\delta}}{H\cos\delta}$ since corresponding lines of the diurnal circle and the equator stand in the ratio of $H\cos\delta : R$ (Vide fig. 20). But this $\perp^{ar}$ is $H\sin\alpha =$

$$R\,\frac{\sqrt{H\sin^2\gamma - H\sin^2\delta}}{H\cos\delta}.$$

If α_1, α_2, α_3 be the Right ascensions of the points on the ecliptic whose modern longitudes are 30°, 60° and 90°, expressed in asus, then α_1, $\alpha_2 - \alpha_1$, $\alpha_3 - \alpha_2$ will give the rising times of the arcs of the ecliptic which stand for Sāyana Meṣa, Sāyana Vriṣabha and Sāyana Mithuna. The rising times of the next three Rasis will be the same in reverse order since Karkata is symmetric with Mithuna with respect to the Equator and similarly Simha and Kanya symmetric with Vriṣabha and Meṣa. The next three are again symmetric with Meṣa, Vriṣabha and Mithuna and the last three with Mithuna, Vriṣabha and Meṣa.

Verse 56. The H cosines of the ends of the Rasis Karkata etc., being multiplied by the radius, and divided by the H cosines of their respective declinations, and the arcs of those H cosines being taken, subtract as before the preceding from the succeeding. Then we have the rising times of the Rasis beginning with Karkataka.

Comm. This is clear from fig. 24. If S be the end of Vriṣabha, S′ in the figure denotes the end of Karkata them from the right-angled triangle SAP,

$$\sin S'A = \sin SA = \sin \widehat{APS} \times \sin PS = \sin A'K \times \cos SK$$

$$\therefore \ \sin A'K = \frac{\sin S'A}{\cos SK}. \quad \text{But } \sin S'A = \cos S' \bumpeq$$

and $\cos SK = \cos \delta$ where δ is the declination at S.

A′K converted into time gives the rising time of AS′ ie. Karkata.

$$\therefore \ \text{the rising time of Karkata} = \frac{\cos S'}{\cos \delta}$$

In tbe Hindu form, it will be

$$\text{Karkata–Rāsi–Udayakāla} = \frac{\text{Karkata–anta–Kotijyā}}{\text{Karkata–anta–Dyujyā}} \times R$$

as stated.

In modern terms $\sin A'K = \cos rK = \dfrac{\cos rS}{\cos \delta}$

from the formula $\cos \lambda = \cos a \cos \delta$. Thus, virtually the formula is a statement of the formula $\cos \lambda = \cos a$ $\cos \delta$. Aslo the formula $\sin AS' = \sin \hat{P} \sin PS$ is parallel to the formula $\sin \delta = \sin \lambda \sin \omega$ which we proved already from Hindu methods.

Verse 57. Still an alternative method.

The H sines of Meṣa etc. being multiplied by H cos ω divided by their respective H cos δ's and the arcs thereof being subtracted as before the preceding from the succeeding we have the rising times of Mesha etc.

Comm. From figure 24,

$SN = SL \cos LSN$ and $\dfrac{SN}{\cos SK}$ = perpendicular from K on

$$\therefore \quad \frac{SL \cos \omega}{\cos \delta} = H \sin rK. \quad SL = H \sin rS$$

$$\therefore \quad H \sin rK = \frac{H \sin rS \times H \cos \omega}{H \cos \delta}$$

We proved the above in a modern way. The Hindu concept is derived from the similarity of SML and ACM′ where C is the centre of the sphere and M′ is the foot of the perpendicular from A on the plane of the equator

$$\therefore \quad \frac{LM}{CM'} = \frac{SL}{CA} \qquad \therefore \quad LM = \frac{H \cos \omega \times H \sin rS}{R}$$

Since LM = SN.

LM divided by H cos δ and multiplied by R gives H sin rK

$$\therefore \quad H \sin rK = \frac{H \cos \omega \times H \sin rS}{R} \times \frac{R}{H \cos \delta} = \frac{H \cos rS \times H \cos \omega}{H \cos \delta}$$

$$\text{i.e. } H \sin \alpha = \frac{H \sin \lambda \times H \cos \omega}{H}$$

Here H cos ω is called Trigṛha-dyu-maurvī because it is the H cosine of the declination of λ when $\lambda = 90°$.

Verses 58, 59. The magnitudes of the rising times.

Those rising times are 1670, 1793, 1937; these in the same and reverse orders diminished or increased by their respective Chara segments which are also in the same and reverse orders give the rising times of the Sāyana Rasis beginning from Meṣa for the locality. The Rasis from Tulā are in a reverse direction i.e. as the Meṣa is projecting upwards above the horizon, Tulā will be projecting below the horizon so that, the time taken by Meṣa to rise is exactly the time taken by Tula to set.

Comm. We shall compute the rising times of Sāyana Rasis for Lanka first i.e. for zero latitude using modern methods from the formula tan α = cos ω tan λ

log tan α=log cos ω+log tan λ; Put λ_1=30, and λ_2 = 60 and take ω = 24°; Let the corresponding α's be α_1, α_2

$$\log \tan \alpha_1 = \log \cos 24 + \log \tan 30^\circ \quad (1)$$

$$\log \tan \alpha_2 = \log \cos 24 + \log \tan 60^\circ \quad (2)$$

$$\log \tan \alpha_1 = 9.9607 + 9.7614 = 9.7221$$

$$\therefore \quad \alpha_1 = 27^\circ - 48'$$

$$\log \tan \alpha_2 = 9.9607 + 10.2386 = 10.1993$$

$$\therefore \quad \alpha_2 = 57^\circ - 42'$$

At the rate of 1 asu for 1′, α_1 = 1668 asus α_2 = 3462; $\alpha_2 - \alpha_1$ = 1794 and since α_3 = the right ascension of 90° Longitude = 90°, α_3 = 5400 so that $\alpha_3 - \alpha_3$ = 1938. These are given by Bhāskara as 1670, 1793, 1937, the first exceeding by 2 asus, the second and third each less by one asu and the total according with the total. The rising times of Karkata etc. will be 1937, 1793, 1670, 1670, 1793, 1937, 1937, 1793, 1670 respectively.

Let us then find the rising times of these Sāyana Rasis at a locality say of latitude 13°. Refer to fig. 21. Let rS represent Meṣa so that the rising times of rE is equal to that of rS. But rE = rA − AE. We have seen rA = 1670 using Bhāskara's value. Sin EA = tan 13° tan δ_1 where δ_1 is the declination of S where rS = 30°. Sin δ_1 = sin 30° sin 24°.

$$\log \sin \delta_1 = 9.6990 + 9.6093 = 9.3083$$

$$\therefore \quad \delta_1 = 11^\circ - 44'$$

$$\therefore \quad \log \sin EA = \log \tan 13^\circ + \log \tan 11^\circ - 44 =$$

$$9{\cdot}3634 + 9.3175 = 8.6809$$

$\therefore$ EA = 2° − 45′ $\therefore$ rE = 1670 − 165 = *1505* asus.

Similarly for λ = 60°, putting α_2, δ_2 in the place of α_1, δ_1 and proceeding as before rE = rA$_1$ − A$_1$E; rA$_1$ = 3463; sin EA$_1$ = tan 13° tan δ_2 sin δ_2 = sin 60° sin 24°

$\therefore \quad \log \sin \delta_2 = 9.6093 + 9.9375 = 9.5468$

$\therefore \quad \delta_2 = 20° - 30'$

$\therefore \quad \log \sin EA_1 = 9.3634 + 9.5758 = 8.9392$

$\therefore \quad EA_1 = 4° - 59' = 299'$

$\therefore \quad rE = 3463 - 299 = 3164$ asus

$\therefore$ Rising time of Sāyana Vriṣabha for the locality $= 3164 - 1505 =$ *1659* asus.

$rA_2 = 5400$, $\sin EA_2 = \tan 13 \tan \delta_2 = \tan 13 \tan 24°$.

$\therefore \quad \log \sin EA_2 = 9.3634 + 9.6486 = 9.0120$

$\therefore \quad EA_2 = 5° - 54' = 354$ asus

$\because \quad rE = 5400 - 354 = 5046$

$\therefore$ Rising time of Mithuna is $5046 - 3164 = 1882$ asus.

Before we proceed to find the rising times of Karkataka, Simha and Kanyā, we shall cast our previous procedure into the Hindu form. In fig. 21, let rS be the Sāyana Mesha. The rising time of rS is measured by rE, because when r is at E, Meṣa is just about to rise and when r is in the position indicated, the extremity of Meṣa namely S is rising. So it means that as rS of the ecliptic has risen, a portion rE of the Equator has risen. As time is measured by the arc of the equator which rises with a uniform speed, we measure the rising time of rS by the arc rE; but $rE = rA - AE$. rA is the Equatorial rising time of rS, because when A is at E, S will be at B i.e. A and S will then be on the equatorial horizon EB simultaneously. Hence $rA =$ *1670* as proved before and stated by Bhāskara. EA is the chara for 30°. The chara for one angula or inch (inch is here used technically, and does not mean what it means in ordinary parlour) as has been stated by Bhāskara and proved by us is 10 Vinadis. (Vide page 181)

But we have taken 13° as our latitude, so that tan ϕ=.2309. Since $\frac{s}{12}$ = tan ϕ = .2309 ∴ s = 2.7708; let us take this as 2.8″ so that the charas 10, 8, 3⅓ found for one inch are to be multiplied by 2.8 to give the local charas. They are in Vinādis, 28, 22.4, 9⅓ or in asus 168, 134·4, 56; for convenience let us take 134.4 as 134, so that the charas are 168, 134, 56. Thus the rising time of Sāyana Meṣa at this locality is 1670 − 168 = 1502 asus i.e. 250 Vinādis = 4–10 Nādis. Then let rS now represent 60°, instead of subtracting EA from rA to get the combined rising time of Meṣa and Vriṣabha, the Hindu practice is to subtract the chara pertaining to Vriṣabha from the equatorial rising of Vrishabha i.e. rE = 1793 − 134 = 1659 as got before. Here it must be noted that EA° is the chara not pertaining to Vriṣabha alone but to Meṣa and Vriṣabha put together. That is why for ease, the chara to Meṣa, the increase in chara for Vriṣabha, and the increase in chara for Mithuna as well as their individual equatorial rising times are given. The increments in the charas are called chara-khandas just as the increments in the H sines are called Jyā-khandas (khands means segments). Similarly the rising time of Sāyana Mithuna is equal to 1937 − 56 = 1881 asus = 313.5 Vinadis = 5–14 Nadis. Now with respect to Karkataka, its equatorial rising time is 1793, for, from fig. 25, the equator at the equatorial place being prime Vertical, if rM be Meṣa, its time of rising is given by rE where E is the foot of the declination circle of M. Similarly if MV represents Vriṣabha, when V comes to the horizon, N the foot of the declination circle comes to the horizon. Thus the rising time of any arc of the ecliptic at an equatorial place is given by the corresponding arc of the equator, which is intercepted between the declination circles of the ends of the arc. So from fig. 26 if r ed ⌢ be the equator, rED. ⌢ the ecliptic, A the Ayana or Summer solstice, P the pole rE, ED, DA etc. the Sāyana Rasis Meṣa etc. a, b, c etc. the feet of the

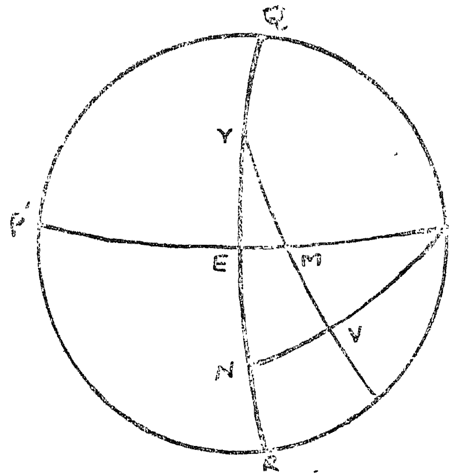


Fig. 25

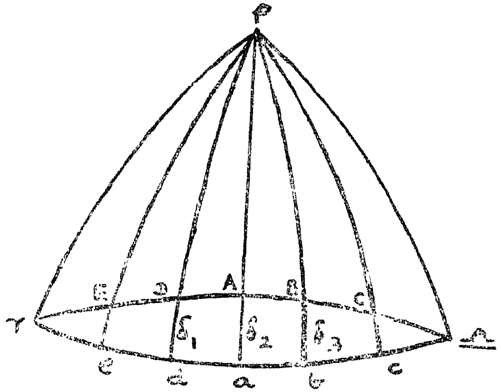


Fig. 26

declination circles of A, B, C etc., since PAE is secondary both to the ecliptic and equator (i.e. perpendicular circle) spherical triangles PAD, PAB are congruent; PBC, PDE are congruent and PC ≏ is congruent with PE *r*. Hence ab = ad i.e. rising times of Karkataka and Mithuna are equal; bc = de i.e. those of Simha and Vriṣabha are

equal and similarly those of Meṣa and Kanyā. It will be noted that the equatorial risings alone are equal in the above cases but not at any other place, for in a place with some latitude when a point like B (fig. 26) comes to the horizon, the foot of its declination circle namely b will not be on the horizon and there arises the chara in between, which has to be taken into account, and be subtracted from or added to the equatorial rising time as the case may be.

Now regarding the chara-khanda of Karkata it is again 56; why it should be so is not proved by Bhāskara but merely stated, nor any commentator took the pains to prove. It can be proved as follows. Let in fig. 26, δ_1, δ_2, δ_3 be the declinations at the ends of Vriṣabha, Mithuna and Karkataka respectively. We know $\delta_1 = \delta_3$. The chara-segments for Mithuna and Karkataka i.e. when Mithuna and Karkataka arc rising are to be proved to be equal, here 56 asus. Their expressions are $\tan\phi \tan\delta_2 - \tan\phi \tan\delta_1$ and $\tan\phi \tan\delta_3 - \tan\phi \tan\delta_2$ i.e. $\tan\phi (\tan\delta_2 - \tan\delta_1)$ and $\tan\phi (\tan\delta_3 - \tan\delta_2)$. Since $\delta_1 = \delta_3$ we perceive that they are equal but of opposite signs. So Bhāskara says rightly "अपचीयमानत्वात् धनम्" i.e. because of negative sign, the chara-segment of Karkataka while being subtracted will be rendered positive'. Hence the rising times of Karkataka, Simha and Kanya will be respectively 1937+56, 1793+134, 1670+168 asus or 1993, 1927, 1838 asus or 332, 321, 306 Vinadis or 5-32, 5-21 and 5-6 nadis. Thus, in as much as the rising times of Meṣa to Kanyā are 1670−168, 1793−134, 1937−56, 1937+56, 1793+134, 1670+168 their total is 30 nadis as should be expected because the equator bisects the ecliptic between ♈ and ≏ and the equatorial interval between ♈ and ≏ is 30 nadis. It will be noted that while the equatorial rising times of Meṣa to Kanyā are symmetrical as 1670, 1793, 1937, 1937, 1793, 1670, their rising times at any other place are not like that but

constitute a different kind of symmetry as 1670−168, 1793−134, 1937−55, 1937+55, 1793+134, 1670+168.

We have now to comment upon the statement "तुलादितोऽमी च विलोमसंस्थाः" which means that the rising times from Tulā to Mīna are in the reverse order i.e. the rising time of Tulā equals that of Kanyā; that of Vrischika equals that of Simha and so on the rising time of Mīna equalling that of Meṣa. Thus the rising times of Tulā to Mīna being in the reverse order are 1670+168, 1793+134, 1937+55, 1937−55, 1793−134, 1670−168 for the aforesaid locality. Why it should be so can be easily seen from the fact that Kanyā and Tulā are symmetric with respect to the line ♈≏ which bisects the ecliptic (Ref. fig. 27). Or again we can see this in another

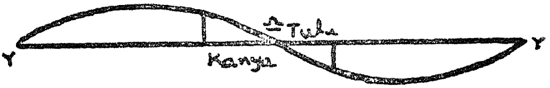


Fig. 27

way; the chara-segments are successively ($\tan\phi\tan\delta_1 - \tan\phi\tan 0$), ($\tan\phi\tan\delta_2 - \tan\phi\tan\delta_1$), ($\tan\phi\tan\omega - \tan\phi\tan\delta_2$), ($\tan\phi\tan\delta_2 - \tan\phi\tan\omega$), $\tan\phi\tan\delta_1 - \tan\phi\tan\delta_2$ ($\tan\phi\tan 0 - \tan\phi\tan\delta_1$), ($\tan\phi\tan\delta_1 - \tan\phi\tan 0$), ($\tan\phi\tan\delta_2 - \tan\phi\tan\delta_1$), ($\tan\phi\tan\omega - \tan\phi\tan\delta_2$), ($\tan\phi\tan\delta_2 - \tan\phi\tan\omega$) ($\tan\phi\tan\delta_1 - \tan\phi\tan\delta_2$), ($\tan\phi\tan 0 - \tan\phi\tan\delta_1$) where δ_1 = declination of 30°, and δ_2 that of 60°. These are there as found before 56, 134, 168, −168, −134, −56 upto Kanyā. But the remaining, though apparently are 56, 134, 168, —168, —134, —56 must be taken with a reverse sign because the rising point of the ecliptic will be to the south of the east point send δ will be negative from 180° to 360° longitude. Hence the chara Segments are 56, 134, 168, —168, —134, −56, −56, −134, —168, 168, 134, 56 so that from Tulā onwards they are in the reverse order as Bhāskara

has stated. Bhāskara's statement yet has another meaning. The ecliptic from 180° to 360 being in the reverse as can be seen in fig. 27, the time by which a particular Rās'i rises, is equal to the time by which its seventh Rās'i or diametrically opposite Rās'i sets. It is important to note that the rising time of a particular Rās'i is not equal to its setting time as can be gauged from the rising times, for, then the rising times of Meṣa and Tulā must be equal which is not the case. The word विलोमसंस्थाः has this meaning as well, rising and setting being reverse directions.

Incidentally there is what is called a handed down i.e. a traditional statement which mentions "मीनमेषौ चतुर्नाड्यः साधं चत्वारिंगोघटौ etc." This clearly pertains to the local rising times of the Sāyana Rās'is at a place of latitude 17°–45′ i.e. in between Rajamundry and Vizianagaram, as in this latitude the chara Segments in asus will be 230, 184, 73 which give rise to such rising times.

We next find what are called Nirayana Swōdayas i.e. the rising times for the locality of the Nirayana Rās'is i.e. the Rās'is from the zero-point of the Hindu zodiac. We shall obtain these magnitudes for Rajahmundry whose latitude is 17°–2ʹ. We shall however take it as 17°; also we shall take the Ayanāms'as to be 21° at present following Varāhamihira. This is one of the contexts where the knowledge of Ayanāms'as is essential. The method is the same as followed before; only, we have to find the declinations, right ascensions, charas and therefrom the rising times of arcs of magnitude 21°, 51°, 81°......351°. Subtracting the preceding rising times from the following we have successively the required rising times. We shall give them under the following table.

Longitude	21°	51°	81	111	141	171	201
Declination	8–13	18–13	23–12	21–51	14–32	3–37	8–13
Right ascension	19–24	48–14	80–12	112–43	143–24	171–44	199–21
Chara (ascentional difference)	2–30	5–41	7–32	7– 3	4–33	1– 6	2–30
Rising time in Vīnādis	169	429	727	1057	1389	1706	2019
Longitude	231	261	291	321	351	21°	
Declination	18– 3	23–12	21–51	14–32	3–37		
Right ascension	228–33	260–12	292–42	323–24	351–44		
Chara	5–41	7– 3	4–43	4–43?	1– 6		
Rising time in Vīnādis	2342	2677	2998	3279	3528		

In modern terms, the rising times of Rasis could be found by solving the spherical triangle γEA (fig. 28) using the Inner side inner angle formula namely

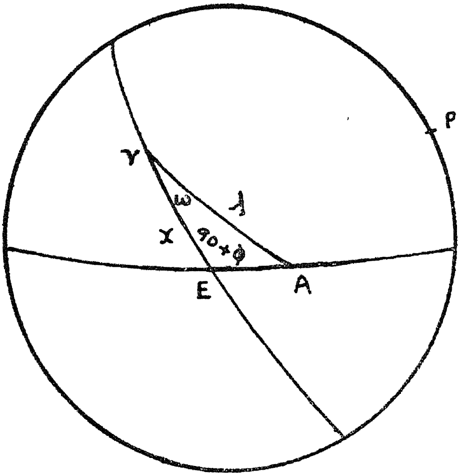


Fig. 28

$\cos x \times \cos \omega = \sin x \cot \lambda + \sin \omega \tan$

$\sin x \cot \lambda - \cos x \cos \omega = - \sin \omega \tan$

putting $\sin x = t$, this reduces to

$t \cot \lambda - \sqrt{1 - t^2} \quad = - \sin \omega \tan$

or $(t \cot \lambda + \sin \omega \tan \phi)^2 = (1 - t^2) \cos$

$\therefore \quad t^2 (\cos^2 \omega + \cot^2 \lambda) + \quad \times \sin \omega \tan \phi \cot \lambda - \cos^2 \omega = 0$

$t =$

$-\sin \omega \tan \quad \cos \lambda \pm \quad \cot^2$

$\cos^2$

Putting successively $\lambda = 30°, 60° \ldots 360°$ and ignoring the negative sign of the radical we have the sines of rising times of arcs of the ecliptic of 30°, 60° etc. Subtracting

the preceding from the succeeding, we have successively the rising times. Thus the rising time of Mesha is

$$\frac{\sqrt{3 \sin^2 \omega \tan^2 \phi + \cos^2 \omega (\cos^2 \omega + 3} - \sqrt{3} \sin \omega \tan \phi}{3 +} \qquad \text{II}$$

That of Meṣa and Vriṣabha put together, the rising time is

$$\frac{\sqrt{\sin^2 \omega \tan^2 \phi + 3 \cos^2 \omega (\cos^2 \omega + \frac{1}{3})} - \sin \omega \tan \phi}{\sqrt{3} (\cos^2 \omega + \frac{1}{3})} \qquad \text{III}$$

The combined rising time of the 3 Rasis Meṣa, Vriṣabha and Mithuna is from I using tables $\cos^{-1} (\tan \omega \tan \phi)$ $= 5046$ asus which exactly accords with what we have found previously namely $1505+1659+1882=5046$. Also putting in I $\lambda = 180°$, we have $\sin x = 0$ or $x = 180° =$ 10800 asus; subtracting 5046, we have the combined times of rising of Karkataka, Simha ahd Kavya to be 5754 asus as we have had. Putting again $\lambda = 270$, we have $\cos^{\circ -1}$ $(\tan \omega \tan \phi) = 2\pi - 5046$ which signifies that the sum of the rising times of the last three rasis is the same as that of the first three which again means that the sum of the rising times of Tula to Dhanus is equal to the sum of the rising times of Karkataka, Simha and Kanya establishing Bhaskara's statement "चिर

Verse 60. Computations of Lagna, Udayāntara and the like from the rising times of big arcs of the ecliptic like Rasis will be approximate, whereas one desirous of greater approximation has to find the same from the rising times of smaller arcs likes Horas and Dṛkkāṇas, so as to be more correct.

Comm. Rasis divided into halves are called horas and if divided into one-third parts are called Dṛkkāṇas. The meaning of the verse is that if after having found the rising time of a particular Rāśi say Meṣa, we say that one-third of that rising time is that of one-third of that Rāśi, we will be making only an appro-

ximate statement just like saying that 'Since 12 Rasis rise in the course of a sidereal day, so each rasi rises in 1800 asus' which is far from truth being based on a crude rule of three. So, Bhāskara says, Acharyas like Aryabhata insisted on finding the rising times of Dṛkkāṇas, in as much, as while computing the Lagna or the point of intersection of the Ecliptic with the horizon, we will be nearer the truth by using the rising times of smaller arcs like Dṛkkāṇas than broadly using those of Rasis.

Verse 61. Bhujāntara correction.

The equation of centre of the Sun multiplied by the equatorial rising time of the Rāsi occupied by the Sun, and divided by 1800, and then again multiplied by the true daily motion of a planet and divided by the number of asus in a day, is the correction in the planet positive or negative according as the equation of centre of the Sun is positive or negative.

Comm. This Bhujāntara correction arising out of the Sun's equation of centre is prescribed even for the Sun, as well as for the other planets. The planets are originally computed for the rising time of the Mean Sun, whereas we are interested to know the positions at the True Sunrise. The position of the Sun also is originally computed for his mean rise and is therefore to be rectified to get his position at his true rise. So even the Sun is not exempt from the correction. It will be noted here that as the equation of centre pertains to the eccentricity of the Sun's orbit, this correction of Bhujāntara is a correction for the so-called modern 'Equation of time due to eccentrity'. In other words, the Sun's equation of centre converted into time is exactly what is called the Equation of time due to eccentricity.

The formula prescribed for the correction is as follows. Suppose the Sun is in a particular Rāśi which rises at the equator in x asus. Let the equation of centre of the Sun be

E minutes of arc. Then the equatorial rising time of E is $\frac{E \times x}{1800}$ asus because there are 1800′ in a Rasi. We have to provide a correction in the planetary position including that of the Sun for this time. If the planet goes Y′ during 21659 asus of a day, what arc is covered by the planet or the Sun in $\frac{E \times x}{1800}$ asus? The result is $\frac{E \times x}{1800} \times \frac{y}{216}$ minutes of arc. This correction is positive if the equation of centre of the Sun is positive, for, we want the planetary position for a latter time than Mean Sunrise, since the positive equation of centre advances the True Sun over the Mean. This correction will be appreciable only in the case of the Moon having a rapid motion.

Verses 62, 63. The correction known as Udayāntara.

The difference in minutes of arc in the longitude of the Sāyana mean Sun and the asus in his Right ascension, multiplied by the daily motion of the planet and divided by 21659 is the result to be added to or subtracted from the planet's longitude according as the asus in the Sun's Right ascension are greater or less than the minutes of arc of his longitude. This is what is called Udayāntara correction in the planetary position.

Comm. (1) We have seen that the Bhujāntara is a correction in the mean planetary position due to the Equation of time in Eccentricity.

(2) This Udayāntara is a correction in the same due to the Equation of time in obliquity.

(3) Some have misconstrued that this Udayāntara correction is the Equation of time in the obliquity itself, whereas it is a correction to be effected in the planetary position due to the Equation of time due to obliquity.

(4) The maximum equation of centre in the Sun has a magnitude $= 13\frac{2^\circ}{3} \times \frac{1}{2\pi} = \frac{41}{3} \times \frac{7}{44} =$
Converting this into time at the rate of 15° per hour (since in one hour diurnal rotation of the earth is equal to 15°) we have $\frac{13}{6} \times 4$ minutes $= \frac{26}{3} = 8'-40''$. The maximum equation of centre according to modern astronomy is $2e$ expressed in radians where $e = \frac{1}{60}(.0167339) =$ [1]
radians $= 2 \times \frac{1}{60} \times \frac{180 \times 7}{22}$ degrees $= \frac{21^\circ}{11} = 1^\circ - 54'$ -
As such the max. Equation of time due to eccentricity is $\frac{21}{11} \times 4' = \frac{84}{11} = 7'-38''$. The small difference in the two values arises out of the difference in the max. equations of centre. Any way it is clear that, in as much as the Bhujāntara correction is necessitated on account of the equation of centre in the Sun, to obtain the planetary positions computed for the mean Sunrise at the time of True Sunrise, the Bhujāntara correction is a correction in *the planetary position*, on account of the equation of time due to eccentricity. The max. correction to be effected in even the quick moving Moon amounts to $\frac{8 \times 790}{24 \times 60} = \frac{79}{18} =$

(5) We have said in the translation of the verse, 'The asus in the Right ascention of the mean Sun', where what exactly is stated by Bhāskara is, the time of rising of the small arc covered by the Sun in the particular Sāyana Rasi in which the Mean Sun is, (भुक्तासव:) together with the rising times of the previous Sāyana Rasis covered by the Sun. The meaning is therefore the rising time of an arc of the ecliptic equal to the Sāyana longitude of the ecliptic which is measured by his Right ascension at the

rate of 15° per hour or 6° per nadi or 10 Vinādis per degree or 60 asus per 60 minutes of arc or as many asus as there are minutes of arc in the right ascension of the mean Sun. So, what is stated by Bhāskara is the difference of the minutes of arc in the mean longitude of the Sun and the minutes of arc in his right ascension ie. $(l-\alpha)$ expressed in minutes. We know that the total equation of time arising out of unequal motion in the true longitude of the Sun ie. $\odot$ in comparision with the equal motion in his right ascension ie. α is measured by $\odot - \alpha$ which may be expressed as $(\odot - l) + (l - \alpha)$ where l is his mean longitude, $\odot - l$ = Equation of centre and so the time expressed by $\odot - l$ is the equation of time due to obliquity. The difference $l - \alpha$ arises out of the obliquity of the ecliptic and so the time expressed by $l - \alpha$ is the equation of time due to obliquity. We have the modern formula

$$\operatorname{Cos}\omega = \frac{\tan \alpha}{\tan l} \text{ so that } \frac{1-\cos\omega}{1+\cos\omega} = \frac{\tan\alpha - \tan l}{\tan\alpha + \tan l} = \frac{\sin \alpha - l}{\sin \alpha + l}$$

$\therefore$ $\operatorname{Sin}(\alpha - l) = \tan^2 \omega/2 \sin(\alpha + l)$. As α is very nearly equal to l, we could write, when expressed in radiaus $\alpha - l = \tan^2 \omega/2 \sin 2l$. Thus the maximum difference between α and l arises when $2l = 90°$ ie. $l = 45$ degrees ie. at the mid-point of the first quadrant; the minimum difference is when $2l=270$ ie. $l=135$ ie. at the middle point of the 2nd quadrant. Also the numerical magnitudes of the max. as well as the minimum value are each $\tan^2 \omega/2$ ie. they are equal. Since this is expressed in radiaus, converting into time the numerical value of the max. and minimum equation of time due to obliquity is $9.87'$. Thus we can write $l - \alpha = 9.87' \sin 2l$. Again where $l = 225$, $2l = 450$ so that $l - \alpha$ will have a positive max.; and again when $l = 315$, $2l = 630$ so that $l - \alpha$ will have a negative maximum value. Also when l has values 0, 90, 180, 270 it is zero. Thus the equation of time due to obliquity is zero at ♈, ie. the vernal equinox; +ve in the first quadrant

increasing from zero to 45 degrees and then decreasing from 45° to 90°; assuming the value zero at 90°, then negative in the second quadrant negatively increasing from zero to a maximum as l increases from 90° to 135°, and then negatively decreasing from the maximum value to zero at the end of the second quadrant; again behaving in the 3rd quadrant as in the first and in the fourth as in the second.

(6) It was stated by Mr. Mazumdar in his introduction to the Siddhānta S'ekhara of S'ripati published by the Calcutta University as well as by pandit Babuaji Misra, the editor thereof that this Udayāntara correction was first mentioned by Sripati, and it meant equation of time due to obliquity. In fact S'ripati states (Verse I ch. elevan) "अन्त्यभ्रमेण गुणिता रविबाहुजीवाऽभीष्टभ्रमेण विहृता, फलकार्मुकेण, बाहोः कलासु रहितास्ववशेषकं ते, यातासवो युगयुजोः पदयोः धनर्णम्"
l expressed in minutes $- \text{H}\sin^{-1}\dfrac{\text{H}\sin \odot\ \text{H}\cos\omega}{\text{H}\cos\delta}$ expressed in asus = what are called elapsed asus and are +ve, +ve, +ve and −ve in the successive quadrants. We saw before that $\text{H}\sin\alpha = \dfrac{\text{H}\sin \odot\ \text{H}\cos\omega}{\text{H}\cos\delta}$ so that S'ripati meant $l - \alpha$, the former expressed in minutes of arc and the latter in asus, or what is the same $(l-\alpha)$ both expressed in minutes or asus. These give the gain of l over α. Immediately after this verse S'ripati goes to a different topic, and never mentions (as understood from the printed text) any further details as to what is to be done with these Yātāsus. Since Bhāskara says explicitly that for these Yātāsus, the planets are to be corrected, we may surmise that there should have been in S'ripati's text also another verse detailing the usage of those asus.

(7) We shall now attend to what Bhāskara gives by way of explanation of the verses in question, The Ahar-

gaṇa found by us is what is called Madhyama Sāvana-ahargaṇa or number of days according to mean solar reckoning, a reckoning made on the basis of taking uniform motion of the Sun in right ascension. The planets computed for the Sun-rise on this basis, are to be corrected for the true Sun-rise which goes according to the measure of ⊙ ie. the true longitude of the Sun. But in the name of Bhujāntara we have effected the correction for $\odot - l$; so what remains is to effect correction for $l - \alpha$. This correction is known as Udayāntara.

In other words, since we could not compute the Sphuta Sāvana-ahargaṇa, when we deem that x days have elapsed according to the mean solar reckoning from the epoch upto the mean Sun-rise $x \pm f$ might have elapsed according to Sphuta Sāvana-ahargaṇa where f is a fraction. A correction is to be effected for this fraction f of a day and it is of the form $\odot - \alpha = \odot - l + l - \alpha$. In the beginning Bhāskara said that mean planets are had being computed out of the mean solar ahargaṇa, at the time when the mean Sun is about the eastern horizon of the equator. Why did he say 'about the horizon'? It is because the mean solar ahargaṇa indicates a Sun-rise on the basis of equal motion in right ascension whereas on the basis of equal motion in l, he would be about the horizon. Had we been able to compute Sphuta-Sāvana-ahargaṇa, we could have got direct the planetary position when the true Sun is on the horizon. The correction for the difference in ⊙ and l having been attended to through Bhujāntara, we are now to attend to the correction for the difference $l - \alpha$.

Verse 64. Another way of looking at the same.

Had we obtained the Ahargaṇa in terms of the local risings of the Rasis in the place of the equatorial, and computed the planetary positions for the local true Sun-rise obtained that way, we would have done the three

corrections namely Bhujāntara, Chara and the Udayāntara as well.

Comm. The correction of chara arises out of the difference between the equatorial risings and local risings of the mean Sun. The local mean Sun pertaing again to uniform motion along the celestial equator Udayāntara has to be effected for $l-\alpha$, ie. for the local mean Sun on the ecliptic. Thus we have obtained the planetary positions for the local mean Sun-rise and to obtain them for the True Sun-rise, Bhujāntara is to be effected.

Verse 65. An alternative method of effecting the Udayāntara correction.

Double the H sine of the Sāyana longitude of the Sun derived out of the smaller H sin-table, being multiplied by the daily motion of the planet and divided by 270 and the result in seconds of arc is to be corrected in the planetary position positive or negative according as the Sun is in the even or odd quadrants.

Comm. (1) In symbols the correction indicated is $\frac{m \text{ H sin } 2l}{270}$ where m is the planet's mean daily motion in minutes of arc, l the longitude of the mean Sun, and H sine is taken where the radius $=120$. In the commentary under the verse, Bhāskara adds 'Each quadrant of the eclipse rises (at the equator) in $\frac{1}{4}$th of a day but each Rāśi does not rise in $\frac{1}{12}$ of a day. Since this Udayāntara correction vanishes when the Sun is at the ends of quadrants it must be construed that this correction increases positively or negatively from the beginning of the quadrant to the middle and decreases from the middle to the end of a quadrant. Saying that a particular Rāśi rises at the equator in n nādikās is only an approximate statement since the Rāśi does not rise uniformly. That is why astronomers like Aryabhata

stipulated finding the risings of smaller arcs like horas and Dṛkkāṇas. Find $\frac{\text{H} \sin l \times \text{H} \cos \omega}{\text{H} \cos \delta}$ ie. H sin α. Take the asus in the arc of this H sine ie. find α in minutes. Then $l-\alpha$ gives the number of asus for which the correction in the planetary position is to be effected, for, by these asus the True Sun-rise is accelerated or belated. In the middle of a quadrant these asus will be a little above 26 Vinadis. To obtain them at any point of the quadrant, take H sin $2l$ as the argument so that the maximum correction will be had at the middle of the quadrant by this argument. Then the rule of three is 'If by 120 as radius, we have 26 Vinadis, what shall we have for H sin $2l$?' The result is $\frac{\text{H} \sin 2l \times 26}{120} = \frac{\text{H} \sin 2l}{4^1}$ approximately. Then another rule of three. "If by 60 Vinadis we have m' of the planetary motion, where the daily motion of the planet is $m°$, what shall we have for $\frac{\text{H} \sin 2l}{4\frac{1}{2}}$?" The result is $\frac{\text{H} \sin 2l \times m}{60 \times 4\frac{1}{2}}$ minutes $= \frac{\text{H} \sin 2l \times m}{270}$. Then the sign of the correction is clear.

(2) We shall now prove Bhāskara's statement that at the middle of the quadrant, the correction is 26 Vinadis. We saw above that $l-\alpha = \tan^2 \omega/2$ H sin $2l$. It is really creditable on the part of Bhāskara to have seen by intuition that the argument is H sin $2l$. The maximum correction is therefore $\tan^2 \omega/2$ expressed in asus or minutes of arc. Let $x = \tan^2 \omega/2$

$\log x = 2 \log \tan \omega/2$. Take $\omega = 24$

Then $\log x = 2 \times \bar{1}.3275 = \bar{2}.6550$

$\therefore \quad x =$ ·04519 radian = 155 minutes of arc ie. 155 asus $= \frac{155}{6} = 26$ apply.

(3) Taking the case of the Moon, the max correction amounts to $\frac{790 \times 120}{270} = 5' - 51''$.

66, 67. The computation of Tithi, Nakṣatra and Yoga.

The elongation of the Moon ie. the excess of the Moon's longitude over that of the Sun in degrees. (In case the former is smaller, add 360° to it and subtract Sun's longitude) being divided by 12 and 6, the quotients represent the elapsed tithis and Karaṇas. If the elapsed Karanas are K, count K − 1 beginning from Bava to get the current Karaṇa and count from S'akuni to get the current one beginning from the mid-moment of Kṛṣnachaturdasī. Take again the planetary position or that of the Moon in particular as well as the sum of the longitudes of the Sun and the Moon both expressed in minutes of arc and divided by 800. The first quotient gives the elapsed stars ie. the Stars covered by the planet or the Moon; whereas the second quotient gives the elapsed Karaṇas. Then take the remainders in seconds and divide by the respective daily motions in minutes i.e. in the case of the planets or the Moon divide by their respective motions and in the case of yogas divide by the sum of the motions of the Sun and the Moon. Then the results give the times in nādīs as to how much the next nakṣatra or yoga have elapsed. If it be required to find as to how long the next nakṣatra or yoga will last, substract the remainder from 800, and divide by the daily motions in minutes as mentioned above. The results give as to how many nādis beginning from the morning concerned, the next nakṣatra or yoga will last.

Comm. (1) A lunation i.e. the time from the moment of New Moon to the next New Moon is divided into 30 parts called Tithis. Their names are pratipat, Dwitīyā etc. upto the 15th purnima or full Moon and again

pratipat, Dwitīyā etc. upto the 30th i.e. Amāvāsyā or New Moon. Thus pratipat starts when ☾ = ⊙ i.e. when the longitude of the Moon is equal to that of the Sun i.e. from the moment of New Moon when the Moon is in conjunction with the Sun. अमा सह=वसतः सूर्याचन्द्रमसावस्या मित्यमावस्या i.e. Amāvāsyā is that point of time when the Sun and the Moon are together. Pratipat lasts till ☾ = ⊙ + 12°; then Dwitīyā begins and lasts till ☾ = ⊙ + 24; Thus purnima begins from the moment when ☾ = ⊙ + 168 and lasts till ☾ = ⊙ + 180° and Amāvasyā begins when ☾ = ⊙ + 348 and lasts till ☾ is again equal to ⊙ Thus a tithi is the time that is taken by the Moon to overcome the Sun by 12° beginning from the moment of New Moon. In other words the tithi is a measure of the phase of the Moon with a particular convention.

(2) In modern astronomy the phase of the Moon or that of a planet is measured by the formula $\frac{1 + \cos EPS}{2}$ where P is the planet or the Moon. Since ES is almost

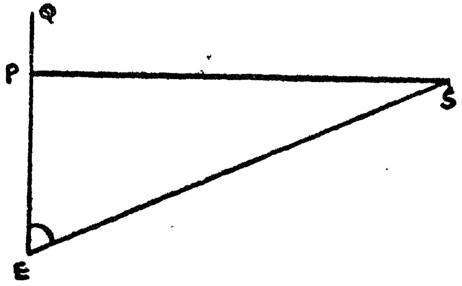


Fig. 29

parallel to PS (Fig. 29) $\widehat{EPS}$ may be taken to be nearly equal to $\widehat{PES}$ which is the elogation of the planet or the Moon so that phase = $\frac{1 - \cos EPS}{2}$ approximately. So

in modern astronomy also the phase is measured through the elongation. But the maximum phase in modern astronomy is taken to be unity as could be seen from the formula, for, when $\widehat{EPS} = 180°$, the phase $= 1$. Thus the phase multiplied by 15 gives approximately the tithi. We shall have occasion to deal with this topic later.

(3) Suppose ☾ − ⊙ = E°. The number of the tithis elapsed is equal to the quotient in $\frac{E}{12}$. Take the remainder $r°$. It means during the current tithi which has a duration of 12°, $r°$ have elapsed. Let the daily motions of the Moon and the Sun be m and s so that the Moon overtakes the Sun by $m - s$ (expressed in minutes for convenience) during 60 nādis. The elapsed time of the current tithi in nādis is given by $\frac{r \times 60 \times 60}{(m - s)}$. But $r \times 60 \times 60$ = Seconds of arc of the remainder so that it is said "गतैष्यविलिप्तिका:". If it be required to find how many nādīs the next tithi lasts, $(12-r)°$ or $(12-r) \times 60'$ is to be gained by the Moon over the Sun. So the time in

$$\text{Nādīs} = \frac{(12 - r) \times 60 \times 60}{m - s} = \frac{\text{एष्यविलिप्तिका:}}{m - s}.$$

(4) A lunation is again divided into 60 Karaṇas and so 6° of increase in elongation correspond to a Karaṇa. These Karaṇas are eleven in number, out of which 4 are fixed to occur during the latter half of Kṛṣna Chauturdas'ī, during the two halves of Amāvāsyā and the first half of the S'ukla pratipat. Their names are S'akuni, Chatuṣpāt, Nāgava, and Kimstughna. So there remain 56 halves of tithis during the lunation during which the remaining seven Karaṇas, Known as Bava, Bālava etc. rotate eight times. Thus the first half of the duration of any tithi is covered by one Karaṇa and the latter by another. The computations of Karaṇas proceeds as with respect to tithis

but dividing the elongation by 6, we are asked to count K − 1, (where K is the quotient) beginning with Bava because the first half of S'ukla pratipat is covered by the fixed Karaṇa Kimstughna. The computation of the elapsed as well as the remaining nādīkas of a particular Karaṇa, it proceeds on the same lines as that of a tithi.

(5) The Nakṣatra in which a planet or the Moon is situated is then found. The zodiac is divided into 27 equal divisions beginning with its zero point and each division is named after the brilliant star of that division. Those stars are As'winī, Bharaṇī etc. The star occupied by the Moon has a special significance in the Hindu calendar, it being spoken of that every day is presided over by a nakṣatra. This happens so because the Moon's sidereal period is approximately 27 days.

(6) To compute the Nakṣatra in which the planet or Moon is, take its longitude in minutes of arc λ; divide by 800, since each star division consists of $\frac{360°}{27} = \frac{360 \times 60}{27}$ minutes = 200′. The quotient gives the elapsed nakṣatras. To get the elapsed nādikās of the current nakṣatra or the remaining, let the remainder be r'. Then the proportion is 'If during the day of 60 nādis, the planet or the Moon goes p minutes what time does it take to cover r' or $800-r'$. The answer is

$$\frac{r' \times 60}{p'} \text{ or } \frac{(800-r') \times 60}{p'} - \qquad p' \quad .$$

(7) Regarding yogas, let the longitudes of the Sun and the Moon be $\odot$ and λ in minutes of arc; let their daily motions be s and m in minutes. If the sum of the longitudes is 800′, we say the first yoga named Viṣkambha is over, if the sum is 1600′, the second Yoga, named prīti has elapsed. Thus going on if the sum is 360°, we say the 27 yogas have elapsed and the first again begins. Since the sum of the daily motions of the Sun and the Moon

are on the average 59′−8″+790′−35″=850′, to cover one Yoga, they take roughly one day since the duration of a yoga is of 800′. To get the number of elapsed yogas, divide ($\odot$+M)′ where M is the longitude of the Moon by (s+m). To get the elapsed nādīs which are given by the remainder r' or the remaining nādīs of the current yoga which are given by 800; the proportion is 'If by $s+m$ gain in the sum of the longitudes we have 60 nādīs, what time is indicated by r' or $800-r'$?'. The answer is

$$\frac{r' \times 60}{m+s} \text{ or } \frac{(800-r)' \times 60}{m+s} = \frac{\text{गतैष्यविलिप्तिकाः}}{m+s}.$$

By the word गतैष्यविलिप्तिकाः is meant therefore the number of seconds of arc as many as the remainder r or $(800-r)$ is in minutes or what is the same the remainder or $800-r$ converted into seconds.

(8) The tithi, karaṇa, nakṣatra of the Moon and yoga constitute four of the Angas of the panchānga or the Hindu Calendar the fifth being the week-day. All these five are supposed to have their effects good or bad on living beings.

Verses 68, 69. The correction what is called Nata-karma.

The Zenith distances of the Sun and the Moon at the end of Purṇimā or Amāvāsyā at the time of lunar or solar eclipse, being expressed in nādis, are multiplied by 6 to get the degrees. Let their H sine be got from the short table of H sines. Multiply it by the equations of centre of the Sun and the Moon. Divide by 4920, and 4361 respectively. If the Sun be in the Eastern hemisphere, let the result pertaining to the Sun be subtracted from his position; if in the Western, let it be added to his position. If the Moon be in the Eastern hemisphere and if his equation of centre be negative let the result be added to his position; if the equation of centre be positive, let the result be subtracted from his position in either of the hemispheres.

Again from these positions, let the tithi be computed and again let the above process be carried out until a constant time is arrived at for conjunction or opposition.

Comm. (1) The above procedure is accepted by Bhaskara as Āgama enunciated by Brahmagupta and reiterated by Chaturveda as giving results that accorded with observation. We shall see that the correction known in modern astronomy as 'correction due to astronomical refraction' is indicated here, though it was not stated explicitly.

In fact what was stated by Brahmagupta was that the periphery of the Manda epicycle given as $13\frac{2}{3}°$ for the Sun and as 31–36 for the Moon hold good only on the meridian but the periphery of the Sun is to be increased or decreased by 20′ according as the equation of centre is negative and the Sun is the Eastern equatorial horizon, or western equatorial horizon. If the equation of centre be positive the reverse correction is to be effected in the periphery i.e. for negative equation of centre.

Periphery on the Eastern equatorial horizon	$=14°-0$
On the meridian	$=13°-40'$
On the Western equatorial horizon	$=13°-20'$
For positive equation of centre	
Periphery on the Eastern unmandala i.e. equatorial horizon	$=13°-20'$
On the meridian	$=13°-40'$
On the Western unmandala	$=14°-0$

In the case of the Moon, for negative equation of centre.

On the Eastern unmandala	= 30 – 44
On the meridian	= 31 – 36
On the Western unmandala	= 32 – 28

For positive Equation of centre.

On the Eastern unmandala	= 30 − 44
On the meridian	= 31 − 36
On the Western unmandala	= 30 − 44

In between the meridian and the unmandala, proportion is to be used. If by H sine Z equal to R, there is a difference of 20′ in the periphery of the Sun, what will it be for an arbitrary H sin Z ? The result is $\text{H sin Z} \times \frac{1}{3} \times \frac{1}{120} = \frac{\text{H sin Z}}{360}$. Then again another proportion "If by $13\frac{2}{3}°$ we have the equation of centre E_1 what shall we have for the above difference ?"

$$\text{The result is } \frac{\text{H sin Z}}{360} \times \frac{E_1 \times 3}{41} = \frac{E_1 \text{ H sin Z}}{4920}$$

$$\text{Similarly for the Moon } - \frac{E_2 \text{ H sin Z} \times 52}{60 \times 120 \times \text{[illegible]}}$$

$$= \frac{E_2 \times \text{H sin Z} \times 52 \times 5}{60 \times 120 \times 158} = \frac{E_2 \text{ H sin Z}}{56880/13} = \frac{E_2 \text{ H sin Z}}{4376}$$

which is taken as 4361 on account of approximating

$$\frac{52}{60 \times 120} = \frac{1}{138} \text{ and then multiplying by } \frac{5}{158}$$

$$= \frac{1}{138} \times \frac{1}{158} = \frac{5}{21804} = \frac{1}{2180 \times \frac{4}{5}} = \frac{1}{4361}.$$

(2) In modern astronomy on account of astronomical refraction a celestial body is elevated towards the zenith by the formula K·tan Z where K has a particular value for all celestial bodies. Thus a body in the Eastern hemisphere, getting elevated the correction is to be negative and in the Western it is to be positive. We shall see how far the given correction accords with this.

(3) Consider for the Sun first for negative equation of centre.

(a) On the east, the negative equation being increased, elevation is effected.

(b) On the west, the negative equation of centre being rendered less, elevation is effected.

(c) For positive equation of centre, on the east it being lessened, again elevation is effected.

(d) For +ve equation of centre on the Western side it being increased again it is elevated.

Thus with respect to the Sun, the stipulated correction effects elevation in all the cases which accords with the effect of the modern refraction.

Then let us consider the case of the Moon.

(e) In the case of negative equation of centre, it being lessened in the East, the effect is depression.

(f) And being increased in the west, the effect is depression.

(g) In the case of positive equation of centre, it being reduced in the east, elevation is effected.

(h) And in the west, it being reduced, elevation is effected.

Thus in the case of the Moon, the phenomenon is recorded as depression in the case of negative equation of centre. Out of these two cases again the equation of centre being lessened is truly desirable as the Hindu equation of centre is in excess of the true value. So it need not be interpreted as depression. Regarding the other case, the error might have been due to the fact that a lesser parallax being taken, whose effect is to depress

the celestial body, depression might have been noticed during the course of an eclipse wherein conjunction or opposition had to be belated; or again the greater equation of centre as was postulated for the Sun, than what it should be when his equation of centre was negative might have depressed the Sun, so that the Moon had to be depressed to arrive at the correct moment of conjunction or opposition. Thus this correction of Natakarma which was accepted by Bhāskara on the reported Āgama of Brahmagupta and also on the right endorsement of Chaturveda, must have been in fact no other than the effect of the phenomenon of astronomical refraction, and what further strengthens this observation is the prescription of H sin Z which is proportional tan Z. Also the modern formula being A tan Z where $A = 58''$ approximately, when Z is sufficiently large, the magnitudes given by Brahmagupta are of the same order as that of the moderns. This correction of Natakarma really reflects much credit on the ancient Hindu observations.

Verse 70. Computing the planetary position for a given moment.

The daily motion of the planet being multiplied by the time that has elapsed or that is to elapse at which the planetary position is to be found, and divided by 60, and the result being substracted from or added to the planetary position found, will render the position hold good for the moment in question. The Sun and the Moon will become by this process of what is called तात्कालिकीव equal up to minutes for the moment of conjunction or opposition. For opposition only the Rās'is differ whereas the degrees, minutes and seconds in their positions will be equal whereas for opposition, the positions are equal in all respects ie. Rās'is, degrees, minutes and seconds too.

Comm. The meaning is clear.

Verses 71 to 75. Obtaining what are called Sukṣma-nakṣatras.

The computation of the nakṣatras done as prescribed before, is only approximate. Now I shall give the method of obtaining what are called Sukṣma-nakṣatras as prescribed by the Riṣis that are required to note auspicious occasions regarding marriages, journeys etc. people who knew about it, told that the six stars Vis'ākha, Punarvasu, Rohiṇī, and the three Uttaras or Uttaraphalgunī, Uttarāṣādhā, Uttarabhadrā have the duration of one and half stars ie. $3/2 \times 790' - 35'' = 1185' - 52''$. The six stars Āslesha, Ārdrā, Swātī, Bharaṇī, Jyeshthā and S'atabhishak have half the duration of a star ie. $395' - 17''$. The remaining 15 alone have one nakṣatra duration ie. $790' - 35''$. A star's duration is the mean daily motion of the Moon ie. $790' - 35''$. The sum total of all the above 27 stars being subtracted from 360°, give the duration of the star what is called Abhijit which occurs after Uttarāṣādha and before S'ravaṇa. To obtain the star in which a planet is situated, convert its longitude in minutes of arc and substract the durations of the stars from As'wini as many as could be substracted. The number of stars whose durations are thus substracted are deemed to have elapsed. The remainder is called the gata or elapsed portion of the current star and the difference of this gata and the duration of the current nakṣatra is called the Ēṣya, ie. unelapsed portion. To obtain the elapsed time or the unelapsed time of the current star the gata or the ēṣya is to be multiplied by 60, and divided by the daily motion of the planet concerned the result being in nādīs.

Comm. One line in the verse 72 is evidently missing which should name Rohiṇī and the three Uttaras. We are able to know them from Bhāskara's commentary as well as from Brahmagupta and S'rīpati. In the course of the commentary Bhāskara reiterates what was stated

by Brahmagupta, that Risis like Pulis'a, Vasistha and Garga spoke about these Suksmanaksatras. The duration of Abhijit calculated as directed is 254′ – 18″. The computation as directed is easy for understanding. The reason for the durations indicated is not clear but is to be taken as based on Astrology.

Verses 76, 77. The duration of the planets' transit into new Rās'is and the duration of the interval between successive stars, tithis, Karaṇas and yogas.

The disc of the planet multiplied by 60, and divided by its daily motion gives the nādīs of transit of the planet from Rās'i to Rās'i. This duration is considered to be holy for performing Vedic rites. It is the holiest with respect to the Sun's transit in particular. A planet in its transit gives partly holy results not so much as the Sun, depending upon the nature of the previous and succeeding Rās'is. The duration of Sandhi for tithis is got by dividing the disc of the Moon expressed in seconds by the difference of the daily motions of the Moon and the Sun; so also with respect to Karaṇas. The Sandhi between two Nakṣatras is obtained by the same measure of the disc of the Moon expressed in seconds of arc being divided by the Moon's daily motion. The Sandhi between two yogas is got by dividing the same numerator by the sum of the daily motions of the Sun and the Moon.

Comm. (1) The Sandhi is the period that elapses during the transit of the disc concerned between the Rās'is and nakṣatra divisions. With respect to tithi, Karaṇa and yoga, the divisions are imaginary not being seen in the Sky and the disc concerned is that of the Moon, and not that of the Sun though the Sun's motion is also taken into account. We say a transit from a division to another is current so long as the disc lies partly in the previous and partly in the latter. So the Sandhi begins when the disc touches the next division and ends when its hind part

touches the previous division. In other words it is the interval between the first contact of the disc with the succeeding division and the last contact with the previous division.

(2) In the case of the Sun the duration of the Sanskrānti is equal to $\frac{32'\frac{1}{2}}{59-2}$ of a day taking the maximum magnitude of the disc and the minimum daily motion approximately $= \frac{65}{2 \times 57} \times 60$ nādīs $= \frac{650}{19} = 34 - 12$ nādīs approximately or more accurately $\frac{32\frac{21}{40}}{56\frac{11}{12}}$

$$= \frac{1301}{40} \times \frac{12}{683} = \frac{3903}{6830} \text{ of a day}$$

$$= \frac{3903}{683} \times 6 \text{ nādīs} = \frac{23418}{683} = 34-17 \text{ nādikās.}$$

Taking liberal boundaries, the वृद्धकारिका or the elders saying is विंशतिः पूर्वे, विंशतिः परे ie. 40 nādikās on the whole.

Here ends the Spaṣṭāshikāra.

ॐ

THE TRIPRAS'NĀDHIKĀRA

Verse 1. The purport of this chapter.

Pandits say that this is the science of time in as much as, herein there is described the method of knowing the direction and the point of space (where a celestial body is situated) given the time. Hence I expound that chapter, which gives that knowledge and which abounds in very important statements, which forms the quintessence of the science of astronomy.

Comm. This chapter is called Tripras'nādhikāra, since this deals with the three questions pertaining to the direction and the point of space of a celestial body for a given time ie. dealing with Des'a, Dik and Kāla. In this chapter we come across the Hindu methods of spherical trigonometry, and gnomonics or S'ankuvedha or observations with the help of a gnomon. Also we find herein a usage of what is called 'Golayantra' or the armillary sphere, which helped the Hindu astronomers to solve all diurnal problems. We find herein Bhāskara excelling himself. This chapter abounds in a good number of technical terms and without a knowledge of this chapter, no one could call himself a Hindu astronomer.

Verses 2–4. To compute what is called lagna given the time.

The lagna or the ascendant as it is called or the point of intersection of the ecliptic with the horizon at a given point of time is obtained as follows. Obtain the Sāyana longitude of the Sun at the point of time at which it is required to find the lagna. Supposing the Sun is in the rth degree of a particular Rās'i, the number of asus which

give the rising time of the arc of $(30 - r)^\circ$ of that Rās'i are called the Bhogyāsus, or the asus which are taken by the remainder of the Rās'i to rise at the place. They are equal to. $\frac{(30 - r) \times T}{30}$ where T gives the asus of the rising time of that Rās'i. The Bhuktāsus on the other hand are the asus which pertain to the rising time of the arc of the first r° of that Rās'is, which are therefore equal to $\frac{r \times \mathcal{T}}{30}$; from the given time substract the Bhogyāsus formulated above; then substract also the rising times of as many subsequent Rās'is as could be substracted. Let 'R' be the remainder of the time given. If t be the rising time in asus of the next Rās'i, $\frac{R \times 30^\circ}{t}$ gives the number of degrees by which the lagna has advanced in the next Rāsi. These degrees added to the previous Rās'is beginning from ν the equinoctial point give the Sāyana or the modern longitude of the lagna. From this Sāyana longitude if we substract the Ayanamsa, we have the Nirayana or the Hindu longitude measured from the Hindu Zero-point of the ecliptic. If, however, the given time after Sun-rise expressed in asus say 'a' falls short of the Bhogyāsus defined above, then, $\frac{a \times 30}{\mathcal{T}}$ where $\mathcal{T}$ is the rising time in asus of the Rās'i in which the Sun is situated, added to the longitude of the Sun, gives the Sāyana longitude of the lagna.

Comm. The substance of these verses, though appears to be simple, yet is complicated which can be better understood with the help of a figure (Ref. fig. 30).

Let SEN be the horizon, νER the celestial equator and νAL the ecliptic where L is the point of lagna. Required to find νL the Sāyana longitude of L from which if Ayanamsa be substracted we get the Hindu

longitude. Let νA, AB, BC, CD, DF be the successive Sāyana Rās'is called Sāyana Meṣa, Sāyana Vriṣabha etc. (The Nirayana Rās'is also starting from the Hindu zero-point are called Nirayana Meṣa, Nirayana Vriṣabha etc. and if in Hindu Astronomy we use the words simply as Meṣa, Vriṣabha etc. especially in panchāngas ie. the Hindu almanacs, we have to construe them as belonging to the Nirayana or the Hindu system whose zero-point is called the first point of the constellation Aswini and not r).

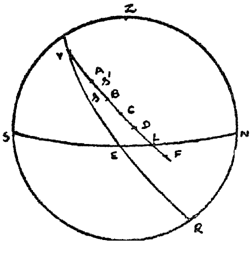


Fig. 30

$rL = rD + DL$ = an integral number of Rās'is, say, 'n' of them ie. $n \times 30° + DL$, where DL is the arc of the Rās'i carrying the Lagna L. The question then resolves itself into knowing how many Rās'is precede 'D' from r and what the measure of DL is. The data are (1) the Nirayana longitude of the Sun as computed by the methods of Hindu astronomy (2) The Ayanamsa of the year ie. 'ro' where o is the Hindu zero-point of the ecliptic. (3) The time after Sun-rise at the place, given in Sāvana units at which we are required to find the Lagna.

Finding the Lagna at a given point of time at a given place is not only necessary in astronomy but its importance is more in what is called horary astrology or Muhūrta S'āstra, whose purpose is to fix an auspicious moment for the performance of marriages etc. as well as in astrology in casting a horoscope.

The finding of the rising times of the various Rās'is at the equator, as well as at a given place was dealt with in the previous chapter. Those rising times are found in sidereal units, a sidereal day being divided into 60 nādis, or 60 × 60 Vinādis or 60 × 60 × 6 asus. These units are of constant magnitude since a sidereal day, which is

the period of diurnal rotation of the earth is of constant magnitude. In the data given above, the time is normally given in Sāvana units, ie. mean solar units. A Sāvana day is of $60 \times 60 \times 6 + 59$ asus, which is greater than a sidereal day by 59 asus because the mean Sun advance by $59' - 8''$ per day among stars and as such the Sun-rise the next day is belated by 59 asus approximately. The given time in Sāvana units could be converted into sidereal units as per the approximate ratio 21600 : 21660 which means for every Sāvana nādi we have to add one asu or for every hour four seconds. Then the procedure of finding the Lagna will be a little different from what it would be if we proceed with the Sāvana units. The complexity mentioned before, arises out of the Sāvana units, and this has been explained by Bhāskara in Golādhyāya under the title तात्कालिकीकरणवासना in the beginning of the chapter called त्रिप्रश्नवासना. Now the procedure will be explained. Let (Fig. 30) S′ be the position of the Sun at the Sun-rise and S his position at the time at which the Lagna is to be found so thst the Sun has advanced by S′S ie. which measured in minutes of arc is called gati-kalās. In a mean solar day these gatis-kalas would be 59, and in the given time after Sun-rise they will be proportional. The time given after Sun-rise pertains to the rising time of the arc S′L which we have in sidereal units. We shall first find the Lagna using sidereal units by measuring S′L, so that we may latter understand Bhāskara's reasoning for his stipulation of
on which basis he finds the Lagna with the Sāvana units taking the position of the Sun at S instead of S′.

We know the position of the sun at sun-rise ie. S′, so that the rising time of As′ ie. the previous arc in the Rāsi in which the Sun is situated is given by Bhuktasus defined above and the rising time of S′B the remaining arc of the Rāsi is given by Bhogyāsus. Subtract from the given time converted into sidereal units if they are not

sidereal, the rising time of S′B ie. the Bhogyāsus, whose formula is $\frac{S'B \text{ (indegrees)} \times T}{30}$ where T is the rising of that Rāsi expressed in asus. Then subtract the rising times of as many subsequent Rās′is as could be subtracted ie. here from the figure the rising times of BC, CD. Then there remains the rising time pertaining to the arc DL from which we could calculate the magnitude of DL in degrees by the formula $\frac{R \times 30}{T}$ where 'R' is the remainder in time after subtracting the rising times of s'B, BC, CD and T the rising time of the Rās′i D.F. Then the longitude of L is rA + AB + BC + CD + DL = 4 × 30 + DL (In the figure shown rD cannot equal 4 Rās′is but will be far less than that but for illustration alone, we have represented it as consisting of 4 Rās′is).

The method of finding the longitude of L as above is quite plain being done on the basis of sidereal units and taking the position S′ of the Sun at Sun-rise. But Bhāskara adopts the position S and Sāvana units which compelled him to take pains to explain what is called तात्कालिकीकरण or obtaining the position S from the Sun-rise position S′. In fact the Sun is at S at the time at which the Lagna is to be found and we have to find DL as before. The argument advanced by Bhāskara is that the rising times of SB, BC, CD, DL measured in sidereal asus, will be just equal to the Sāvana asus, from Sun-rise, because the arcs S′L and SL differ by S′S ie. by the gatikalas pertaining to the time after Sun-rise. In other words. if the rising times of S′A + AB + BC + CD + DL in sidereal asus give the sidereal time after Sun-rise, the rising times of SA + AB + BC + CD + DL in the same sidereal asus give the Sāvana time after Sun-rise. Hence Bhāskara used the word तात्कालिकार्कैण in the beginning of the verse 2. Then he himself raised a purvapakṣa or

the argument of an imaginary opponent namely "Is the time given measured after Sun-rise Sāvana (mean solar) or Nākṣatra (ie. sidereal)? If it be the former, how is it you are subtracting the rising times of *s*A, AB etc. which are sidereal from your Sāvana units? Further, should you not take the position of the Sun at Sun-rise namely S′, because the time given is what has elapsed after Sun-rise? Also, why should you complicate matters by accepting Sāvana units when the question is simple if dealt with sidereal units?

To this Bhāskara answers as follows –

"True it is, what you say. Generally in day to day life, time is given only in Sāvana units and not sidereal. Further you cannot avoid Sāvana measure, for, in the case of an arc moved by a planet, in its diurnal circle time is measured in the Sāvana units pertaining to the planet. (The Sāvana units of a planet are different from what they are for the Sun depending upon the arc moved by the planet in question during a day). These Sāvana units are what are termed Kshetra-Vibhāgātmika or what depend upon the arc covered in the diurnal path in contradistinction to the Kāla-Vibhāgatmika units or sidereal units. (In other words pure time is what is measured in sidereal units which is a standard measure, whereas time which has the bias of the motion of the planet also ie. which we seek to measure by the arc moved by a planet in its diurnal path, is Kṣetra-Vibhāgātmika). Thus having had to accept the Sāvana measure also, we seek to proceed on that basis, though we could convert the Sāvana measure into the sidereal and proceed without complication". The argument which Bhāskara gives for तात्कालिकीकरण is further as follows. The measure of the arc SL using the rising times in asus ie. sidereal units, is the Sāvana measure of the arc S′L done in sidereal units. Instead of subtracting the rising time of S′A from the given time converted into

sidereal, we subtract the rising time of SA from the given Sāvana time and the result will be the same, for,

$-SA = -(S'A - S'S) = S'S - S'A$. which means subtracting SA from the time tantamounts to increasing by S′S and subtracting S′A. This increase by S′S is adding what have been defined as gati-kalas so that automatically the Sāvana units got converted into sidereal units, by taking the position S and subtracting SA instead of taking the position S′ and subtracting S′A. The result is the same. So, it is said 'तात्कालिकार्कस्य' in the beginning of the verse 2.

In the second case mentioned in verse 4, if the time given, falls short of the Bhogyāsus, then simply $\frac{x \times 30}{T}$ where x is the time given in asus and T the rising time of the Rāśi in which both the Sun and the lagna are then situated gives the arc in degrees which if added to the longitude of the Sun gives that of the lagna. Here also the position 'S' counts.

Verses 5 to 6½. To find conversely the time that has elapsed after Sun-rise given the lagna.

The Bhogyasus of the Sun and the Bhuktāsus of the Lagna together with the rising times of intermediate Rasis gives the time required.

If the Sun and the Lagna both be in the same Rāśi, then the arc in between them, multiplied by T and divide by 30, gives the time required.

If, however. the longitude of the Lagna falls short of that of the Sun, ie. if the Sun be below the horizon, (in this case SL > 180°) then finding the time of rising of SL and subtracting from a day, we have the time of the Lagna before Sun-rise.

However, here. there is one complication if we consider the तात्कालिकार्क ie. s the Sun at the given time. This position of s cannot be had unless we know the time, which is itself required. So we get in the first place the time pertaining to S′L, which is the time measured in sidereal units the position S′ being that of the Sun at Sun-rise. If we don't take recourse to convert this time in sidereal units to Sāvana units using the proportion between them, then the alternative is to obtain the position S using the time obtained and then calculate the time again in Sāvana units.

If the time given to find the Lagna, be sidereal, it goes without saying that we find it from S′. Also S′ being given and if the time after sun-rise is required in sidereal units for a given Lagna, the method of successive approximation is unnecessary.

Verse 7. To find the Lagna before Sun-rise called Vilōmalagna.

Suppose it be required to find the lagna before Sun-rise, given the time before Sun-rise. Obtain the then position of the Sun and find his Bhuktāsus; subtract them from the given time; from the remainder, subtract the rising times of as many Rāśis as could be, rāśis behind the Sun's position. If R be the remainder in the time after these subtractions, then $R \times 30/T$ where T is the rising time of is the next preceding Rāśi, together with $n \times 30$, where n the number of integral Rāśis subtracted and the arc of the next Rāśi by which the Sun has advanced in his Rāsi at the time of the Lagna (known from the position of S found) the sum total of these three items being subtracted from the position of S gives the longitude of L.

Comm. Easy.

Verse 8. To obtain the East-West line.

The East-West line is roughly the join of the extremities of the morning shadow as well as that in the after-noon of a guomon placed at the centre of a circle drawn on a plane with any arbitrary radius, when those shadows equal the radius of the circle. But this line is to be deflected keeping its western point ie. the extremity of the morning shadow fixed through a distance $\frac{K(\sin\delta_1 \sim \sin\delta_2)}{\cos\phi}$ at the eastern end perpendicular to it, where the above distance is measured in units, which measure K.

Comm. The east and west points are where the celestial equator cuts the horizon. The east point is thus the point where the Sun-rises when he is exactly at the vernal equinox. The question is how to draw the east-west line on a plane. For this we are asked to draw a circle with any radius on that plane. The plane is described here as अम्भःसुसमीकृतक्षिति that kind of surface as is determined by the surface of water there. Such a kind of surface forms approximately a horizontal plane not of course exactly because such a surface is really spherical, the earth being a sphere. But because the radius of the earth is sufficiently large, we can take such a surface to be a horizontal plane for all practical purposes. Having drawn a circle place the gnomon vertical at the centre. In the morning note the extremity of the shadow when it equals the radius. In the afternoon also mark the point when the shadow equals the radius. Join those two points. It represents rougbly the east-west line, roughly because the Sun's declination changes in between the two moments however small the change might be. Ignoring the change in the declination, this line will be east-west because of the following reason. The length of the shadow is 12 tan z where 12 units are the measure of the gnomon. But since the shadow on both the occasions is equal to the radius of the circle z, the zenith-distance

will be the same on both the occasions. Then the spherical triangles PZS_1 and PZS_2 where P is the celestial pole, Z the Zenith and S_1 and S_2 are the positions of the Sun on the two occasion, are congruent their three sides being respectively equal, provided we take $PS_1 = PS_2$ ie. $90 - \delta_1 = 90 - \delta_2$ ie. $\delta_1 = \delta_2$ on both the occasions. When the two triangles are thus congruent, $P\hat{Z}S_1 = P\hat{Z}S_2$ ie. S_1 and S_2 are equidistant from the plane of the Prime-Vertical. Hence the extremities of the shadows will be equidistant from the East-West line; or this may be seen in another way; $S_1 S_2$ will be perpendicular to the meridiān plane and as such parallel to the plane of the Prime-Vertical.

The correction mentioned in the verse is known as the Agrāntara correction which was originally given by Chaturvedā chārya and then accepted by Srīpati. Why it is called Agrāntara is because it is a change in what are called Karnavrittāgras of the two occasions where we shall see in due course that the formula for Karnavrittāgra is $\frac{K \sin \delta}{\cos \varphi}$ *where K is hypotense of the gnomonic triangle formed* by the gnomon and its shadow S at any place and time. This correction is a very minute correction and as a matter of fact could be ignored. But the fact that the correction was cognized and correctly formulated testifies to the knowledge of the sphere which the above acharyas had. Assuming the formula of the Karnāgra here (it will be proved by us later in this chapter) if δ_1 and δ_2 be the declinations on the two occasions respectively the Karnāgras will be $\frac{K \sin \delta_1}{\cos \varphi}$ and $\frac{K \sin \delta_2}{\cos \varphi}$, K being equal on the two occasions because the shadows are equal. Hence the correction being the difference of the Agrās, it is $\frac{K (\sin \delta_1 \sim \sin \delta_2)}{\cos \varphi}$ as stated by Bhās-

We shall now prove it in modern terms from the spherical triangle PZS. We have the formula $\sin \delta = \sin \phi \cos z + \cos \phi \sin z \sin a$ where PZS = 90 − a, a being the Hindu azimuth measured from the East point. Multiply the above equation by K and divide throughout by $\cos \phi$ where K is called the Chāyākarṇa equal to $\sqrt{12^2 + S^2}$, S being the gnomonic shadow at the moment. Hence

$$\frac{K \sin \delta}{\cos \phi} = K \cos z \tan \phi + K \sin z \sin a \; (1)$$

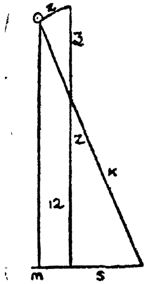


Fig. 31

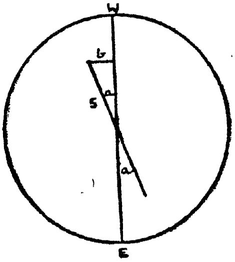


Fig. 32

But from Fig. 31, $K \cos z = 12$, $K \sin z = S$, (2) and from Fig. 32, $S \sin a = b$ where b is called the Chāyābhuja ie. Chāyābhuja = $K \sin z \sin a$ (3). Thus

But again from Fig. 33, when ⊙ the Sun at vernal equinox is on the meridian and as such has a meridian zenith-distance equal to ϕ, $12 \tan \phi = s$ where s is called the Vishuvat-chāyā or equinoctial shadow. Thus we have

$$\frac{K \sin \delta}{\cos \phi} = s + b \; (4).$$

Again if the Sun be on the horizon, from Fig. 34, E ⊙ is called the Agrā, A, so that from the triangle P

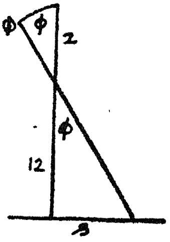


Fig. 33

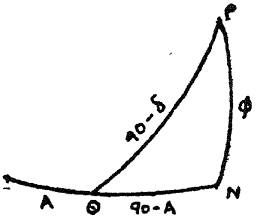


Fig. 34

$\cos(90 - \delta) = \cos\phi \cos(90 - A)$ ie. $\sin \quad = \sin A$ or

in the Hindu form $H \sin A = R \frac{\sin}{\cos} \quad =$ Agrajyā (5).

This Agrajyā is in a circle of radius R, and if it be reduced to a circle whose radius is K, is will be $\frac{K}{R} \times \frac{R \sin\delta}{\cos\phi}$ $= \frac{K \sin\delta}{\cos\phi}$ *which is called Karnāgrā.* Hence we have *Karnāgrā* $= s + b$ which we shall write as $a = b + s$ (6). This is an important formula which is going to be formulated later in verses 72, 73. In the above formula, s being constant, by differentiating $\delta a = \delta b$ which means that the variation in the bhuja is on account of the variation in the Karnāgrā. If in Fig. 35, Cω', CE$''$ be morning and evening shadows when they are equal as per the verse under comment, M$\omega' = b$ the morning bhuja, NE$'' = b'$, the evening bhuja dE$'$ is the variation in the bhuja ie. $b - b' = \delta b$ which is formulated and equal to δa. But $\delta a = \delta\left(\frac{K \sin\delta}{\cos\phi}\right) = \frac{K\,\delta\,(\sin\delta)}{\cos\phi}$, ϕ being constant and K also being constant because the shadow S is constant and $K = \sqrt{S^2 + 12^2}$ = constant on both the occasions. $\therefore\ \delta b = \delta K = \frac{K}{\cos\phi}(\sin\delta_1 - \sin\delta_2)$ as stated by Bhāskara.

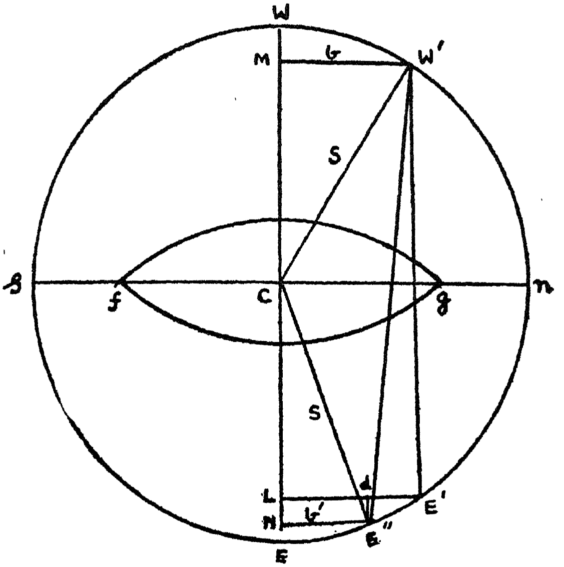


Fig. 35

9. Having determined the East-West line as mentioned, by drawing with a compass with the same radius on either sides of Eω (Eω is the East-West line through the centre of the circle drawn parallel to the rectified East-West line namely E′ ω′ where E″ ω′ is the approximate East-West line obtained by the join of the extremities of the shadows) two arcs which intersecting each other form a fish-like enclosure as shown in the figure and as such called Matsya meaning a fish, and joining the ends of the fish namely *f*, *g* and producing *fg*, we have the south and the north points *s* and *n*. Or again, the north-south line could be had from Fig. 36, where EC is a rod held in the direction of the north-pole as seen by the eye at *e* and EA and CB the lines indicated by the plumb-line called अवलम्बकसूत्र A and B being on the plane and AB joined passing through N, the north-point. Or again the directions could be determined as follows from a single shadow S namely Cω′ in Fig. 35,

calculating the bhuja $\omega'M$ and Koti CM; Bhuja and Koti being known, and the shadow $C\omega'$ being drawn, holding two rods whose lengths are equal to the bhuja and Koti perpendicular to each other, one extremity of the bhuja-rod being held at ω' and one extremity of the Koti-rod being held at C, and the rods making a right angle at M, then the bhuja-rod determines the north-south direction and the Koti-rod the East-West direction.

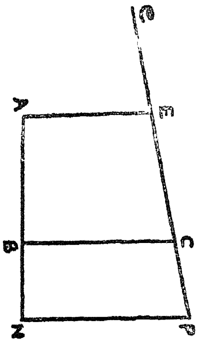


Fig. 36

Verse 10. The Bhuja is defined as the distance of the extremity of the shadow from the east-west line where the S'anku or gnomon is placed at the intersection of $E\omega$ and NS. Koti = $\sqrt{a^2 - b^2}$ ∴ Koti will be in the East-West direction. So Chayā-koti = K sin z cos a (7).

Comm. Easy.

Verse 11. The Chayākarna, K is equal to $\sqrt{S^2 + 12^2}$, so that $\sqrt{K^2 - 12^2} = S$ or $\sqrt{(K + 12)(K - 12)} = S$.

Comm. Easy.

Verse 12. The S'anku is also called Nara or Nā. The zenith-distance of the Sun at Noon when the Sun is in *r* is the latitude of the place, called pala or Aksha; the altitude then is called lamba or colatitude.

Comm. The word S'anku we have previously used for the gnomon. It is also used for the H cosine of the zenith-distance and to differentiate it from the previous S'anku called Dwādasāṅgul'a-S'anku or twelve-unit-length S'anku, it is termed Mahā S'anku and occasionally Iṣta-S'anku. Mahā S'anku or *Iṣta S'anku* = *H cos z* (8). Thus in figure (37) ⊙M = H cos z. In the fig. where *gn* = gnomon, *go* = S

the shadow, $\odot$ = the position of the Sun whose zenith-distance is $\odot z$. If z' be taken as the zenith $\odot z'$ measures the zenith-distance whereas if z be taken as the zenith $\odot z$ is the zenith-distance. The apparent inconsistency that both $\odot z'$ and $\odot z$ are taken as the zenith-distance is not there if we consider the zenith-distance as the angle $\odot \hat{n} z' = \odot \hat{o} z$. $O \odot = R$, $\odot z = z$, so that $\odot L = H \sin z$, and $LO = \odot M = H \cos z$ = S'anku or Nara or Nā. it is called Nara or Nā which means man, because a man may consider himself as a gnomon, which is called S'anku. So the word is also applied to the parallel $H \cos z$ parallel to the gnomon and called Mahās'anku. *H sin z is called Drigjyā* (9) zenith-distance is known as Nati because it is depression from the zenith. $\odot K$ (Fig. 37) is called the unnati or altitude.

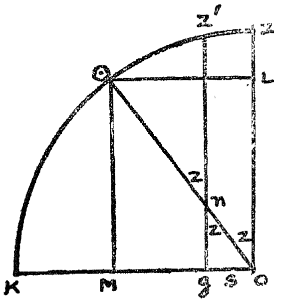


Fig. 37

Verses 13 *to* 17. Latitudinal triangles (Ref. Fig. 21).

(1) The right-angled triangle formed by the gnomon, the equinoctical shadow and the hypotenuse called Vishuvat-karna is the fundamental latitudinal triangle, which is like knowledge that will be the basis of all good things of the world, for example, respect, money, fame and happiness (Fig. 33).

(2) The second latitudinal triangle is that which is formed by $H \sin \phi$, $H \cos \phi$ and the radius R of the sphere (Ref. $\triangle$ O Q L Fig. 38).

(3) The third latitudinal triangle is that formed by Kshitijyā, $S_1 B_1$ Krānrijyā $E_1 B_1$ and Agrajyā $E_1 S_1$ ie. the projected triangle $E_1 S_1 B_1$ of ESB (Fig. 21) on the

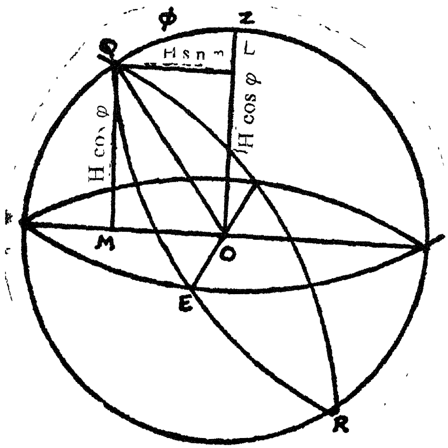


Fig. 38

plane of the meridian - where is the centre of the sphere.

(4) The fourth latitudinal triangle is $E_1 S_1 F_1$ the projection of ESF (Fig. 21) on the meridian plane where $E_1 F_1$ is the Sama-S'anku or the S'anku of the celestial body when it is on the prime vertical. $E_1 S_1$ is the Agrajyā as mentioned, $S_1 F_1$ is what is called Taddṛti.

(5) The fifth latitudinal triangle is $E_1 B_1 F_1$ where $E_1 B_1$ is Krāntijyā or H sin δ, $E_1 F_1$ is the Sama-S'anku defined above and $B_1 F_1$ is what is called the higher segment of the Taddṛti which is equal to Taddṛti minus Kujyā or Kshitijyā.

(6) The sixth latitudinal triangle is $E_1 D_1 B_1$ where $E_1 D_1$ is called the first segment of Agrajyā, $D_1 B_1$ is what is called un-mandala S'anku or H cos z of the celestial body when it is on the unmandala or the Equatorial horizon and $E_1 B_1$ Krāntijyā.

(7) The seventh latitudinal triangle is $D_1 S_1 B_1$ where $D_1 S_1$ is the second segment of the Agrajyā, $S_1 B_1$ is the Kujyā, and $D_1 B_1$ is the unmandala S'anku.

(8) The eighth latitudinal triangle is $B_1 L_1 F_1$ where $B_1 L_1$ is equal to the first segment of Agrajyā, $L_1 F_1$ is the higher segment of Agrajyā, $L_1 F_1$ is the higher segment of the Sama-Vritta-S'anku, and F_1B_1 is the upper segment of Taddṛti mentioned before.

Comm. It was already mentioned that a latitudinal triangle is such a right-angled tri-angle constituted by the chords of the celestial sphere where the angles in the triangle are ϕ, $90 - \phi$, 90°. The side opposite to φ is called the Bhuja, that opposite to $(90 - \phi)$ is called Koti and the third Karna. Such triangles are not only eight as have been mentioned, but many more will be there as mentioned by Bhāskara. They are all formed as mentioned by him by the intersections of the diurnal paths and the celestial equator with the circles of the sphere namely horizon, prime vertical, meridian, Equatorial horizon and declination circles. These circles clearly intersect at ϕ or $90 - \phi$. The eight triangles mentioned are those whose elements will be entering computations. There is another important latitudinal triangle with which we have to deal later namely that formed by what is called Hriti, S'anku, and S'ankutala (Fig. 39).

O_3K = Agrā ; O_3C = S'anku-tala ; CN = S'anku-bhuja = CK $\therefore$ Agrā = S'ankutala + S'anku-bhuja

Agrā $= R \dfrac{H \sin \delta}{H \cos \phi}$; S'ankutala $= H \cos Z \tan \phi$

S'anku-bhuja $= \dfrac{H \sin z \, H \sin a}{R}$ (Ref. fig. 40′)

$$\therefore \quad R \frac{H \sin \delta}{H \cos \phi} = H \cos z \tan \phi + \frac{H \sin z \, H \sin a}{R}$$

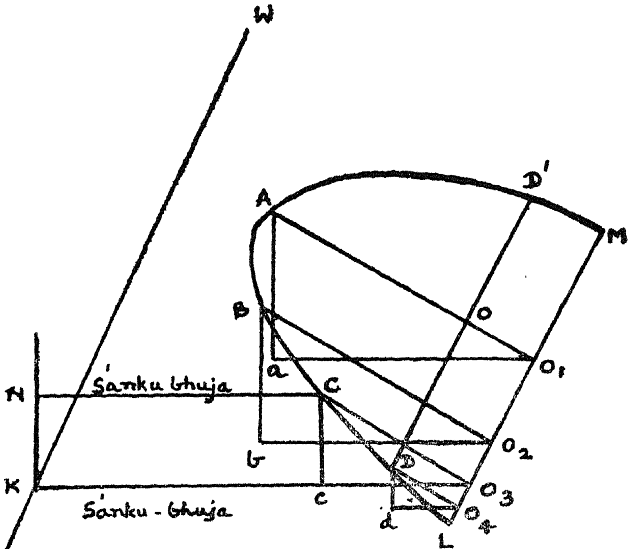


Fig. 39

or in modern terms $\sin \delta = \sin \phi \cos z + \cos \phi \sin z \sin a$ as derived from the triangle PZS. Formula I is with respect to the Mahā S'anku ⊙ L of fig. 40; if it be reduced to the S'anku of the gnomon the S'anku-bhuja ⊙ N becomes the Chāyābhuja pr which will be equal to

$$\frac{\text{H sin z H sin a}}{\text{R}} \times \frac{\text{K}}{\text{R}} = \text{K sin z sin a whereas,}$$

Agrā becomes $\frac{\text{R H sin}}{\text{H cos } \phi} \wedge \frac{\text{K}}{\text{R}} = \frac{\text{K sin}}{\text{cos}}$ which is called

Karnāgrā and S'anku-tala becomes

$$\frac{\text{H cos z tan } \phi \times \text{K}}{\text{R}} = \text{K cos} \qquad = 12 \tan \phi =$$

which is a constant quantity. It will be noted that Karnāgra differs from point to point since K differs from time to time during a day. In fact Karnāgra is the perpendicular dropped from the extremity of the shadow

on the line parallel to the East-West line and north of it at a distance of the equinoctial shadow. As the shadow varies from time to time Karnāgra varies from time to time. Chayābhujā is the perpendicular dropped from the same extremity of the shadow on the East-West line. (Vide figure under verses 72, 73).

Let fig. 39 represent the diurnal path of a celestial body which is parallel to the Equator. Let A be the point where the celestial body culminates or crosses the meridian of the place, B the point where it crosses the prime vertical, C any arbitrary point of the orbit and D the point where it is on the Equatorial horizon, unmandala. Drop perpendiculars from A, B, C, D to the plane of the horizon, namely Aa, Bb, Cc, Dd. Let $MO_1\ O_2\ O_3\ O_4\ L$ be the line of intersection of the plane of the diurnal circle with the horizon so that, LM is called the Udayāstasūtra, L being the point where the body rises and M where it sets. A line through D, the point where the diurnal path cuts the Equatorial horizon, drawn parallel Eω the East-West line will be a diameter of the diurnal circle and as such bisects the path. Draw perpendiculars from a, b, c, d to ML to meet it in O_1, O_2, O_3, O_4. By the theorem of three perpendiculars Ao_1, Bo_2, Co_3, Do_4 will be perpendiculars on ML in the plane of the diurnal circle. It is clear from the figure that all these right-angled triangles Ao_1a, Bo_2b, Co_3c, Do_4d are not only mutually similar but also are similar to the latitudinal triangles, in as much as the angles $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, the angles between the vertical plane and the diurnal plane being equal to the angle between the planes of the prime Vertical and the Equator, are all ϕ and angles $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ are right angles. Hence these triangles are also latitudinal. In fact Bbo_2, Ddo_4 were already included by us in the list of the eight latitudinal triangles since they are congunent to $E_1\ F_1\ S_1$ and $D_1\ B_1\ S_1$. As a matter of fact O_2B is Taddhṛti itself, O_4D Kujyā,

Bb = Sama-S'anku, and Dd unmandala S'anku. O_2b is not actually the Agrajyā but parallel and equal to it, since Agrajyā is the H sine of SE of fig. 21, which is the perpendicular from S on Eω. Similarly do_4 is not the second segment of Agrajyā but a parallel and equal segment. Thus Bb is the perpendicular distance between EE′ and FF′ ie. Eω and a parallel through F to Eω (fig. 21.) Dd is the perpendicular distance between DD′ and BB′; bo_2 is the perpendicular distance between Eω and SS′; do_4 the perpendicular distance between DD′ and SS′ (fig. 21); Bo_2 the perpendicular distance between FF′ and SS′; Do_4 the perpendicular distance between BB′, SS′. In this fig. 39, O_1A is called Hṛti, Co_3 = Ishta-Hṛti or any arbitrary hṛti. Taddhṛti Hṛti and Kujyā are special cases of Ishtahṛti. Hṛti is the maximum of Ishtahṛti. Calling Aa, Bb, Cc, Dd S'ankus in general ao_1, bo_2, co_3, do_4 are called S'ankutalas. Aa is called Dinārdha-S'anku or the S'anku of the mid-day; whereas Dd is the unmandala-S'anku and Bb Sama-S'anku. Cc is called Ishta-S'anku. Perpendiculars from A, B, C, D on the plane of the prime vertical are called S'anku-bhujas. Since B is on the prime-vertical itself, the S'anku-bhuja is zero and at this point BO_2 is Agrajyā. In the arbitrary case at C, the perpendicular from C on the plane of the prime-vertical being S'anku-bhuja, which is equal to the perpendicular from C on Eω, and Co_3 being the S'anku-tala, and since the perpendicular distance between ML and Eω is the Agrajya, which is equal to the sum of O_3C and the S'anku-bhuja S'anku-tala + S'anku-bhuja = Agrajya. (10) which is a different expression of (5). We shall present this analytically. Putting S'anku = H cos z, and using the latitudinality of Cco_3 $\frac{H \cos z}{12} = \frac{Co_3}{s} = \frac{Co_3}{K}$ applying similarity with the first fundamental latitudinal triangle where s = equinoctial shadow, and K the Viṣuvat-Karṇa. Hence we have

$$Co_3 = \text{Ishta-hṛti} = \frac{K}{12}\,H\cos z = \frac{H\cos z}{H\cos\varphi} = \frac{\cos z}{\cos\varphi}\ (11)$$

$$Co_3 = \text{S'anku-tala} = \frac{s}{12}\,H\cos z = H\cos z\tan\varphi$$

$$\frac{H\cos z\ H\sin\varphi}{\quad} = R\cos z\tan\varphi$$

From the triangle PZS, we have the formula $\sin\delta = \sin\varphi\cos z + \cos\varphi\sin z\sin a$ written under verse 8

$\therefore \frac{\delta}{\cos\varphi}\cos z\tan\varphi + \sin z\sin a$ or in the Hindu form

$$\frac{R\sin\delta}{\quad} = H\cos z\tan\varphi + \frac{H\sin z\ H\sin a}{R}.\ \text{I.}$$

We saw under verse 8 that $\frac{R\ \sin\delta}{\cos\varphi} = \frac{RH\sin}{H\cos\varphi}$

= Agrajyā (formula 5). Also we have from (12) above $H\cos z\tan\varphi$ = S'anku-tala. From fig. 40, if SM be drawn secondary to the prime-vertical, the right-angled spherical triangle SMZ gives

$\sin x = \sin z\sin a$ when SM $= x$ or in the Hindu form $H\sin x = \frac{H\sin z\ H\sin a}{R}$; but $H\sin x$ we defined as S'anku-bhuja, so that the equation I above may be written Agrajyā = S'anku-tala + S'anku-bhuja which is (10). *H sin a is called Dik-jyā* and *H sin z*, *Dṛk-jyā.* Formula (10) or (6) is the Hindu expression of the modern formula $\sin\delta = \sin\varphi\cos z + \cos\varphi\sin z\sin a$. *But the beauty lies in reducing* (10) to the formula *derived under verse* 8 *namely* $a = b + s$ ie. formula (6) to the *horizontal plane, introducing* the concepts of Karṇāgra *and Chāyābhuja* (Ref. fig. 40′). The lines corresponding to O_1A and O_3C in the Equatorial plane are called Antyā and Ishtāntya.

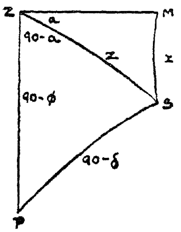


Fig. 40

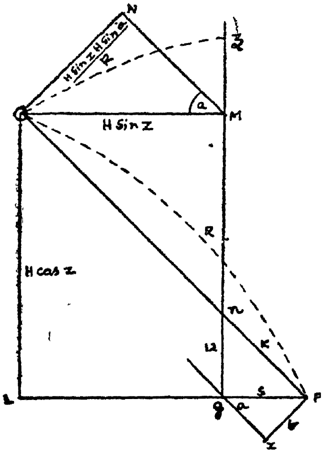


Fig. 40

Hṛti $= O_1A = OO_1 + OA$ = Kujyā + H cos δ or Dhujyā VI since $OO_1 = DO_4$ = Kujyā. The line in the Equatorial plane corresponding to Kujyā is *Charajyā which is equal to R tan φ tan δ* (13) so that we can write Kujyā = R sin δ tan φ = *H sin δ tan φ* (13′). Also from VI

Antyā = R + Charajyā = R + R tan φ tan δ (14) which gives us the duration of half the day where R gives 6 hrs and Charajya the increment in day due to φ and δ.

Verse 18. To obtain the magnitudes of the various chords or the elements of the latitudinal triangles.

The elements or the sides of all these latitudinal triangles are mutually derivable from similarity.

Comm. Easy.

Second half of Verse 18. The radius multiplied by the Kotis or Bhujas and divided by the Karnas gives H cos φ and H sin φ.

Comm. The seven triangles except the second, of the eight latitudinal triangles are compared here with the second. Remembering Bhujas are the sides of the triangles opposite to φ and Kotis opposite to 90 − φ,

$\frac{\text{Bhuja}}{\text{Karṇa}}$ in any triangle $= \frac{\text{Bhuja}}{\text{Karṇa}}$ in the second latitudinal triangle $= \frac{\text{H sin } \varphi}{\text{R}}$ $\therefore$ $\frac{\text{R} \times \text{Bhuja}}{\text{Karṇa}} = \text{H sin } \varphi$ Akshajyā or Palajyā (15) Similarly $\frac{\text{Koti}}{\text{Karna}}$ in any triangle is equal to the $\frac{\text{Koti}}{\text{Karṇa}}$ in the second latitudinal triangle $= \frac{\text{H cos } \varphi}{\text{R}}$ so that $\frac{\text{R} \times \text{Koti}}{\text{Karṇa}} = \text{H cos } \varphi =$ lambajyā (16).

e. The arcs of H sin φ and H cos φ are respectively the Akshams'as and lambams'as as they are called ie. latitude and colatitude. H sin φ and H cos φ are also obtainable thus $\sqrt{\text{R}^2 - \text{H sin}^2 \varphi} = \text{H cos } \varphi$ and $\text{H sin } \varphi = \sqrt{\text{R}^2 - \text{H cos}^2 \varphi}$ Or again $\frac{\text{H sin } \varphi \times \text{Koti}}{\text{Bhuja}} = \cos \varphi$ and $\frac{\text{H cos } \varphi \times \text{Bhuja}}{\quad} = \text{H sin } \varphi$ where the Bhuja and Koti may belong to any latitudinal triangle.

Verse 20. Agrajyā can be had by multiplying Krāntijyā by Karṇa of any lat. triangle and divided by its Koti. Also Sama-S'anku $= \frac{\text{Karṇa}}{\text{Bhuja}} \times$ Krāntijyā and

$$\frac{\text{Sama-S'anku} \times \text{Karna}}{\text{Koti}} = \text{Taddhṛti}.$$

Comm. The first of these statements pertains to the similarity of the third triangles to the others. The second of the statements pertains to the similarity of the fifth latitudinal triangle to others whereas the third pertains to that between the fourth and the others.

Thus Sama-S'anku or S.S.

$$\frac{H \sin \delta \times R}{H \sin \varphi} \text{ or } \frac{R \sin \delta}{\sin \varphi}$$

$$\text{Taddhṛti} = R\frac{H \sin \delta}{H \sin \varphi} \times \frac{R}{H \cos \varphi} = \frac{R \sin \delta}{\sin \varphi \cos \varphi} \quad (18).$$

Verse 21. $\text{Taddhṛti} = \frac{\text{Karṇa} \times \text{Agrajya}}{}$

Comm. This pertains to the similarity between the fourth latitudinal triangle and the others.

Latter half of Verse 21 *and first half of Verse* 22.

$$\text{Sama - S'anku} = \frac{\text{Taddhṛti} \times \text{Koti}}{\text{Karna}} = \frac{\text{Agrajyā} \times \text{Koti}}{\text{Bhuja}}$$

$$\frac{\text{Sama-S'anku} \times \text{Bhuja}}{} -$$

Comm. The first statement is made out of the similarity between the fourth lat. triangle and the others whereas the second statement and the third as well are made out of the similarity between the third and others.

II half of Verse 22. Sama - S'anku = Uppor

$$\frac{\text{segment of Taddhṛti} \times \text{Karna}}{\text{Koti}}$$

Comm. The similarity is between the fifth triangle and others.

Verse 23. Kujyā = Krāntijyā × Bhuja/Koti

$$\text{Upper segment of Taddhṛti} = \frac{\text{Krāntijyā} \times \text{Koti}}{\text{Bhuja}} \text{ and}$$

Kujyā + Upper segment of Taddhṛti.

Comm. The first statement is through the similarity of the third triangle with others, whereas the second is through the similarity of the fifth with the others. The third statement is clear from Fig. 21.

Verse 24. $\frac{\text{Kujyā} \times \text{Bhuja}}{\text{Karṇa}}$ = second segment of Agrajyā $\frac{\text{Krāntijyā} \times \text{Koti}}{\text{Karṇa}}$ = first segment of Agrajyā and Agrā = Sum of the two segments.

Comm. The first statement is based on the similarity between the seventh triangle and others and the second on the similarity between the 6th and the others.

Verse 25. $\frac{\text{First segment of Agra} \times \text{Bhuja}}{\text{Koti}}$

= Un-mandala-S'anku and $\frac{\text{Krantijyā} \times \text{Bhuja}}{\text{Karṇa}}$

= Un-mandala-S'anku.

Comm. Both the statements are based upon the similarity of the sixth triangle and others. Thus un-mandala s'anku = U.S. = $\frac{\text{H} \sin \delta \times \text{H} \sin \varphi}{\text{R}}$ =

$$= \text{R} \sin \delta \sin \varphi.$$

Verse 26. $\frac{\text{First segment of Agrā} \times \text{Koti}}{\text{Bhūja}}$

= Un-mandala-S'anku = Kujyā × Koti/Karṇa.

Comm. The first statement is based on the similarity of the sixth latitudinal triangle and others and second that between the seventh and others.

Sama-s'anku — unmandala s'anku = upper segment of Sama-s'anku.

Verse 27. $\frac{\text{Agrā} \times \text{Bhuja}}{\text{Karṇa}}$ = Kujyā

Taddhṛti — Kujyā = upper segment of Taddhṛti.

Comm. The first statement is based on the similarity between the third triangle and the others and the second statement is evident.

Second half of verse 27. Other elements could be derived from what is already known and from what has been obtained. Also by alternando and invertendo we could pass from one element to the other and vice-versa.

Verse 28.
$$\text{Karṇa} = \sqrt{\text{Bhuja}^2 + \text{Koti}^2}$$
$$\text{Bhuja} = \sqrt{\text{Karṇa}^2 - \text{Koti}^2}$$
$$\text{Koti} = \sqrt{\text{Karṇa}^2 - \text{Bhuja}^2}$$

Thus the third could be had from the other two in all the triangles.

Verse. There are sixty-three ways of obtaining H sin ϕ and H sin ϕ. On account of hundreds of ways of obtaining Agrajyā etc., there are an infinite number of ways of obtaining H cos ϕ etc.

Comm. Under verse 23 Bhāskara says that there are 98 ways of obtaining Taddhṛti. Taking the third latitude triangle, H sin δ conld be obtained in seven ways, from this H sin δ, Kujyā could be obtained in seven ways; hence, according to the principle of association namely that when one thing could be done in m ways and another in n ways, both the operations could be together performed in mn ways, so Kujyā could be obtained in $7 \times 7 = 49$ ways; similarly the upper segment of Taddhṛti could be had in 49 ways; so that adding the two Taddhṛti could be obtained in 98 ways.

Similarly suppose we have to find H sin δ. H cos ϕ could be found in seven ways and from H cos ϕ, H sin δ could be found in seven ways. Thus H sin ϕ could be

found in 7 × 7 ways = 49 ways. From R, H sin φ could be found in seven ways by using similarity with the other seven latitudinal triangles except the second. Also obtaining H cos φ in seven ways and using the formula H sin $\varphi = \sqrt{R^2 - H\cos^2\varphi}$ we have seven more ways. Thus in all there are 7 × 7 + 7 + 7 = 63 ways. Similarly H cos φ could be found in 63 ways. Extending this to Agrajyā etc. which could be in as many or more ways themselves finding H sin φ therefrom means again finding it in 69 × 69 ways and so on. Since there is no end in counting all these ways, it is said that there are infinite ways to find it. The word 'infinite' here connotes only a very large number of ways not exactly what we mean by the word 'infinity'.

Verse 30. To find what is known as Koṇa-S'anku.

As a first approximation take Koṇa-S'anku = $\sqrt{R^2 - 2A^2}$ where A = Agrajyā and Koṇa-S'anku means H cos z when the azimuth is equal to 45°. Then take Agrajyā $\pm$ the above Koṇa-S'anku × $\frac{s}{12}$ = S'anku-bhuja = b (say) then again Koṇa-S'anku = $\sqrt{R^2 - 2b^2}$. Then again take Agrajyā $\pm$ the above Koṇa-S'anku × $\frac{s}{12}$ as the new bhuja and proceeding thus by the method of successive approximations, we arrive at a constant value which gives the Koṇa-S'anku.

Comm. Bhāskara gives later the method of obtaining H cos z ie. the S'anku pertaining to any zenith-distance. So, he need not have given a separate treatment for this Koṇa-S'anku. But in as much as Brahmagupta and other previous writers gave it he has also given the same. He gives here the method of finding the Koṇa-S'anku by the method of successive approximations as given by S'ripati.

We saw before that Agrā = S'anku-tala + bhuja. So, in the first place as a first approximation, Agrā is taken as S'anku-bhuja. Since, when $z = 45°$ the perpendiculars from the celestial body on the planes of the prime-vertical as well as meridian are equal, and since the perpendicular on the plane of the prime-vertical is called S'anku-bhuja = b (say) $2b^2 = H \sin^2 z$. This is so because $H \sin^2 z$ = Sum of the squares of the perpendiculars on the planes of the meridian and prime-vertical. But $H \sin^2 z = R^2 - H \cos^2 z$. $\therefore R^2 - 2b'^2 = H \cos^2 z$. So, taking Agrā as the bhuja b as a first approximation, $\sqrt{R^2 - 2b^2}$ gives us $H \cos z$. From this using formula III under latitudinal triangles namely $H \cos z \times \frac{s}{12}$ = S'anku-tala, obtain the approximate S'anku-tala, from the approximate $H \cos z$ got above. Now using the formula Agrā = S'anku-tala + S'anku-bhuja obtain S'anku-bhuja as Agrā $\pm$ S'anku-tala, where the +ve sign is taken when the Sun has a southern declination, and the difference sign when the declination is north. Taking this S'anku-bhuja, b', Koṇa-S'anku is now $\sqrt{R^2 - 2b'^2}$. In the first place we took the Agrā itself as the bhuja; but here we have a better approximation for the S'anku-bhuja. From this Koṇa-S'anku again, obtain a still better approximation for S'anku-bhuja and proceeding thus till a constant value is obtained, we have the required Koṇa-S'anku. *This is a beautiful example where the method* of successive approximation was used by the Hindu Astronomers to a good advantage. It will be noted here that the S'anku-tala is always treated as extending south ie. the S'anku-tala will be south of the S'anku since India's latitudes are all north. Also it is said that when the Sun has southern declination, $A + S = B$ and when northern $A - S = B$ when A = S'anku-Agrā or simply Agrā (in contradistinction to Karnāgrā reduced to a circle of radius K the chayākarṇa) S = S'anku-tala and B = S'anku-bhuja. This convention

of signs is to be correlated with the modern. We have from the formula derived out of PZS, $A = S + B$. According to modern convention when δ is north, it will be taken to be positive and when a is to the north of the East point it also will be taken to be positive so that, (1) when δ is north and a north, $A = S + B$ ie. $B = A - S$; this accords with the Hindu convention namely 'सौम्येत्वन्तरम्' (2) when δ is south and a south, $-A = S - B$ so that $B = A + S$; this also accords with the Hindu convention, namely याम्ये योग: (3) But, however, when δ is north and a south ie. when the Sun having northern declination comes to the south of the prime-Vertical, $A = S - B$ so that $B = S - A$. This accords with the Hindu convention if only we take $B = | A - S |$ when δ is north.

Bhāskara makes two statements at the end of the commentary under this verse namely that when δ is south and $A > 2431$, there will be no Koṇa-S'anku and that when δ is north and $s > 17'' - 5'''$ there will be four Koṇa-S'ankus. We have to verify these statements. The first statement is evident because H sin (Agrā) > H sin 45° ie. > 2431′, no Koṇa-S'anku will be formed above the horizon because the diurnal path above the horizon will be to the south of the points K, K′ on the horizon where $EK = 45°$ and $\omega K' = 45°$ where E, ω are the east and west points and K and K′ respectively lie on the eastern and western horizons between E and S and ω and S, S being the south point. Regarding the second statement, in order that there may be a Koṇa-S'anku in the north Agrajyā > H sin 45 ie. $\frac{\sin \delta}{\cos \varphi} > \sin 45°$ ie. $\sin \delta > \sin 45°$ $\cos \varphi$. Taking the max. value for δ namely 24°, $\log \sin 24° > \log \sin 45 + \log \cos \varphi$ ie. $9.6093 > 9.8495 + \log' \cos \varphi$ ie. $\log \cos 45 < 9.7598$ $\therefore$ $\varphi > 54° - 54'$ $\therefore$ $\tan \varphi > 1.4229$ $\therefore$ $s = 12 \tan \varphi > 17'' - 4.5'''$ which is taken by Bhaskara as $17'' - 5'''$. For a lesser value of δ, a still greater value of φ will be required

as may be seen by taking $\delta = 20°$, $\varphi > 61° - 6'$. Thus Bhāskara gave the minimum latitude which could enjoy four Koṇa-S'ankus.

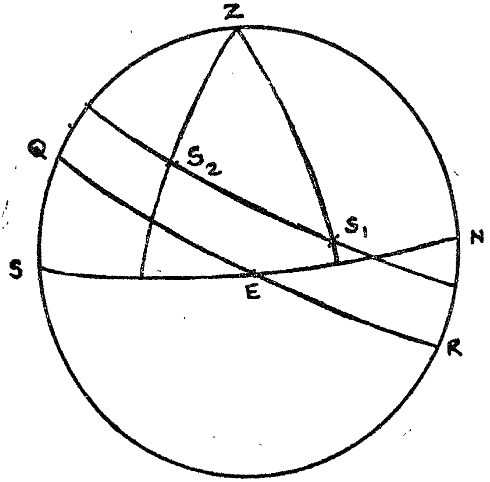


Fig. 41

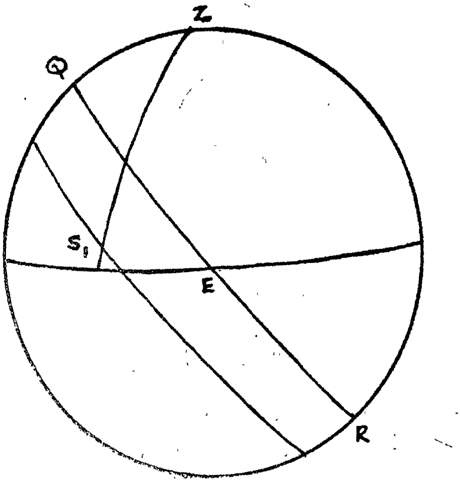


Fig. 42

From fig. 41, it is clear that there are two Koṇa-S'ankus at S_1 and S_2 during the forenoon and similarly two at S_1' and S_2' in the afternoon where S_1' and S_2' are the symmetrical points of S_1 and S_2. From fig. 42, it is clear that if Agrā $<$ H sin 45° when δ is south, there will be one Koṇa-S'anku in the forenoon at S_1 and one in the afternoon at the symmetrical S_1'.

Verses 31 *and* 32. H cos z at noon known as Dinārdha-S'anku.

By 'northern hemisphere' it is meant that the Sun is in the northern hemisphere ie. his S'āyana longitude ie. modern longitude lies between 0° and 180°, and 'the southern hemisphere' means that the Sun's longitude lies between 180° and 360°. The direction of δ may be got from the above convention. The latitude and colatitude are always deemed as south and north respectively.

The latitude and colatitude being 'added to subtracted from or being decreased by' as the case may be, the declination, we have the zenith-distance and the altitude of the celestial body at Noon. The zenith-distance and the altitude are mutually complements.

Comm. In Hindu Astronomy the words "उत्तरगोले" "दक्षिणगोले" are very often used to connote that the Sun is on the north or the south of the celestial equator respectively, so that the declination could be automatically known to be north or south respectively. Regarding the latitude, the peculiarity in Hindu Astronomy is that what we call north latitude in modern astronomy *is construed as south* in as much as the celestial equator gets depressed south in northern latitudes. The colatitude SQ in fig. 41 on the other hand extends north from the south, so that, it is construed as north.

The word 'Samskāra' is used in Hindu Astronomy in the meaning given above in the translation. For

example in the equation $A = S + B$, we say that the Bhuja is had by a Samskāra between A and S. The meaning of Samskāra given by Bhāskara is "समदिशोर्योग: भिन्नदिशोरन्तरम संस्कार:" Latitude being regarded as southern, if the Sun's declination is 12° north and the latitude 20°, then as they are of opposite direction, effecting the Samskāra as directed $20 - 12 = 8$ = zenith-distance (South) = Nata as it is called similarly $70 + 12 = 82$ = Altitude = Unnata ; here we have added because, both lamba and declination are north. Similarly when $\delta = 24°$ north, and $\varphi = 20°$ as before (south)

$24 - 20 = 4°$ = zenith-distance (north) = Nata

$70 + 24 = 94$ = unnata (north). But, we take $180 - 94 = 86°$.

In the above working in the first case we found $\varphi - \delta$, whereas in the second we found $\delta - \varphi$. This difference in treatment is not taken objection to, since, the word Antara is used to take the positive value of the difference alone and so in the first instance the nata is pronounced as south, whereas in the second it is pronounced north.

In modern astronomy, however, we have the formula $z + \delta = \varphi$, considering z as positive if south, δ and φ positive if north. Here $8° + 12° = 20°$ (first case cited above) and $(-4°) + 24° = 20°$ (2nd case, z being negative, for, it is north. In the Hindu symbolism we have to pronounce separately when z is south or north, whereas in modern symbolism the sign alone informs its direction. Similarly in the equation $A = S + B$, we have to pronounce 'north bhuja' or 'south bhuja' as the case may be, whereas having a convention that δ is +ve when north. and also the Hindu azimuth (measured from the East point) the sign of bhuja indicates its direction. In other words we differentiate the two cases $A - S$ and $S - A$ giving them signs and deducing the direction of the bhuja

from the sign itself without an appeal to a picture or without ascertaining whether the northern Agrā prevails over the Southern S'anku-tala or the Southern S'anku-tala prevails over the northern Agrā. Thus the Dinārdha S'anku in symbolism = H cos ($\varphi \pm d$) (20).

Verse 33. Here at noon, Dṛg-jyā is the H sine of nata and the S'anku is H sine of unnata.

Second half of 33 and first half of Verse 34.

The product of R and the unmandala-S'anku divided by Charajyā is called Yaṣṭi. The Yaṣṭi increased by Un-mandala-S'anku gives H cos z according as the Sun is north or south of the equator.

Comm. Unmandala-S'anku is H cos z when the Sun is on the unmandala. From the sixth latitudinal triangle, wherein Unmandala-S'anku is Bhuja and Krāntijyā Karṇa, so by comparing with the second latitudinal triangle (or rather operating with the second triangle to signify the Hindu method).

$$\frac{\text{Krāntijyā} \times \text{Bhuja}}{\text{Karṇa}} = \text{Unmandala-S'anku}$$

$$= \frac{\text{H sin } \delta \times \text{H sin } \varphi}{\text{R}} \text{ (already}$$

We saw before Charajyā = R tan φ tan δ. Hence as directed in the verse $\frac{\text{R} \times \text{H sin } \varphi \text{ H sin } \delta}{\text{R} \times \text{R tan } \varphi \text{ tan } \delta}$ = Yaṣṭi

$$= \frac{\text{H cos } \varphi \text{ H cos } \delta}{\text{R}} \text{ (21).}$$

$$\therefore \text{ H cos z (at Noon)} = \frac{\text{H cos } \varphi \text{ H cos } \delta}{\text{R}} \pm$$

$\frac{\text{H sin } \varphi \text{ H sin } \delta}{\text{R}}$ according as the Sun is on the north or south of the equator.

Hence H cos z (at Noon) = Natajyā

$$\frac{\text{H} \cos \varphi \ \text{H} \cos \delta \pm \text{H} \sin \varphi \ \text{H} \sin \delta}{\text{R}}$$ or in modern symbolism $\cos (\varphi \mp \delta)$ already derived under (20).

We shall now show how the formulation is done by the simple rule of three (Ref. fig. 39). If a parallel through $o_4 d$ is drawn to cut Aa at a_1, then Aa_1 is called the Yaṣti, which is vertical. The triangles Aoa_1 and D$o_4 d$ are similar so that $\frac{\text{A}a_1}{\text{A}o} = \frac{\text{D}d}{\text{D}o_4}$ $\therefore$ $\text{A}a_1 = \frac{\text{DD} \times \text{A}o}{\text{D}e_4}$.

But $\frac{\text{A}o}{\text{D}o_4} = \frac{\text{R}}{\text{Charajyā}}$ for, all the lines of the diurnal circle and the equator stand in the ratio

$$\frac{\text{H} \cos \delta}{\text{R}} = \frac{\text{A}o}{\text{R}} = \frac{\text{D}o_4}{\text{Charajyā}} = \frac{\text{Kujyā}}{\text{Charajyā}} \therefore \frac{\text{A}o}{\text{D}o_4} = \frac{\text{R}}{\text{Charajyā}}$$

$$\therefore \ \text{A}a_1 = \frac{\text{Unmandala S'anku} \times \text{R}}{\text{Charajyā}} = \text{Yaṣti}$$ as formulated.

Now Dinardha S'anku = Aa = Aa_1 + $a_1 a$ = Aa_1 + Dd = Yaṣti + Unmandala S'anku. It is evident from fig. 21 why in the northern sphere the sum is to be taken whereas in the southern, the difference is to be taken.

Latter half of verse 34. Definition of Hṛti and Antyā.

The sum or difference of Dyujyā and Kujya will be similarly Hṛti, whereas the sum or difference of Charajyā and radius will be Antyā.

Comm. We defined formerly Hṛti and Antyā our commentary on the latitudinal triangles. From fig. 8 Hṛti = o_1A = $o_1 o$ + oA = o_4D + oA = Kujya + H cos (Dyujyā) (22).

In the parallel great circle, the Equator we have therefore Antyā = Charajyā + R (already derived).

Verse 35. $\text{Antyā} = \frac{\text{Hṛti} \times R}{H \cos \delta} = \frac{\text{Hṛti} \times \text{Cha}}{\text{Kujyā}}$

$\therefore \text{Hṛti} = \frac{\text{Antyā} \times H \cos \delta}{R} = \frac{\text{Antyā} \times \text{Kujyā}}{\text{Charajyā}}$ by what is called Guṇa-ccheda-Viparyaya ie. alternando.

Comm. Evident.

Verse 36. To obtain Dinārdha-S'anku from Antyā and Hṛti

$\frac{\text{Antyā} \times \text{un-mandala S'anku}}{\text{Charajyā}} = \frac{\text{Hṛti} \times 12}{K} =$ S'anku.

Comm. The second formula is derived from the similarity of △ Ao_1a (fig.) with the first latitudinal triangle. From the similarity of Aao_4, Ddo_4 fig. 39,

$Ao./Do_4 = \frac{Aa}{Dd}$ $\therefore Aa = \frac{\text{Hṛti} \times \text{unmandala-S'anku}}{\text{Kujyā}}$

But Aa = Dinārdha-S'anku

$\therefore$ Dinārdha S'anku = $\frac{\text{Hṛti} \times \text{U.S.}}{\text{Kujyā}}$ (U. S. = Unmandala-S'anku.)

But Kujyā Charajyā

D.S. (Dinārdha-S'anku) = $\frac{\text{Charajyā}}{\text{Antyā}} \times$ U.S.

i × 12

Verse 37. Meridian Zenith distance.

$\text{Agrā} \pm \frac{\text{Hṛti} \times \text{bhuja of a lat. triangle}}{\text{Karṇa of a lat. triangle}} =$

where z is the meridian zenith distance.

Comm. This formula is a special case of the formula $A = S + B$ since the H sine of the meridian zenith distance is the S'anku-Bhuja at noon. From fig. 39

$$\frac{\text{Hṛti} \times \text{bhuja of a latitudinal triangle}}{\text{Karṇa of a latitudinal triangle}} = O_1a = \text{Dinārdha}$$

S'ankutala.

The operation of sign has been already explained.

Verse 38. An alternative method.

The meridian zenith distance of the Sun can be had also by the formula (Hṛti $\pm$ Taddhṛti) $\frac{\text{B.L.T.}}{\text{K.L.T.}}$ where B.L.T. and K.L.T. are the bhuja and karṇa of any latitudinal triangle.

Comm. (Ref. figures 43 and 21). Let E_1 be the centre of the armillary sphere so that QE_1R is the diameter of the celestial equator which is on the median plane. Let $S_1 F_1 S$ be the diameter of the diurnal circle of the Sun, which is also on the meridian plane so that $S_1 F_1$ is the Taddhṛti, $S_1 S$ is the Hṛti and S the position of the Sun on the meridian.

H sin z = SM = $SF_1 \sin \hat{F_1} = SF_1 \sin \phi =$ (Hṛti $-$ Taddhṛti) $\times \frac{s}{K}$ where s/K can be replaced by $\frac{\text{B.L.T.}}{\text{K.L.T.}}$ (s=equinoctical shadow and k the Viṣuvat-Karṇa).

In the Southern sphere, H sin z = $S'N = S'F_1' \sin \phi$ $= (S'S_2 + S_2F_1') \sin \phi =$ (Hṛti + Taddhṛti) $\times \frac{s}{k}$.

Verse 39. Still another way of obtaining the *m. z. d.* (meridian-zenith-distance).

R $-$ H versin (altitude) = H sin z

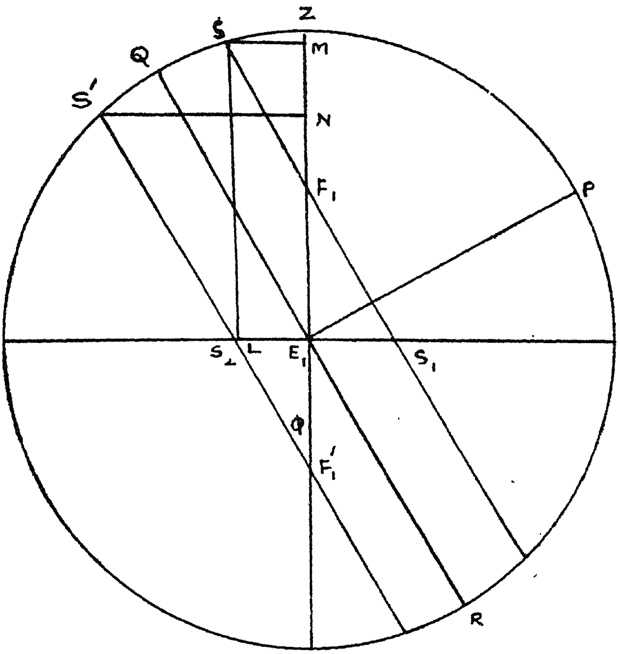


Fig. 43

This formula gives not only the meridian zenith distance but H sine of the zenith-distances of the Sun when he is on the Koṇa-Vṛtta or prime-vertical or unmandala.

Comm. From fig. 43, H sin z = SM = sL = R — H versinc (S*s*) = R — H versine (altitude) as given. Since R — H versine (altitude) = R — {R — H cos (90 — z)} = R — (R — R sin z) = H sin z, so this formula applies wherever the Sun be. This is almost begging the question as H sine of z is being sought through H versine of (90 — z).

First half of the verse 40. To obtain the shadow S and K the Chayakarṇa of any shadow

$$\frac{\text{H sin z} \times 12}{\text{H cos z}} = \text{S and } \frac{\text{R} \times 12}{\text{H cos z}} = \text{K}.$$

Comm. (Ref. fig. 44).

$$\frac{12.\ \text{H sin z}}{\text{H cos z}} = 12 \tan z = \text{S}. \qquad \text{Also } \frac{12}{\text{K}} = \cos z = \frac{\text{H cos z}}{\text{R}}$$

so that $\frac{12\ \text{R}}{\text{H cos z}} = \text{K}.$

The Hindu method of looking at this through the similarity of $\triangle$S OM$\odot$ and Ogn the gnomonic triangle. is as fellows. $\odot$M is called Mahā-S'anku ie. H cos z ; $\odot$L is Dṛkjya or

$$\text{H sin z} = \text{OM}.\ \frac{12}{\text{H cos z}} = \frac{\text{S}}{\text{H sin z}} \text{ so that S} =$$

12 H sin z/H cos z. Also,

$$\frac{\quad}{\text{O}\odot} = \frac{12}{\text{H cos z}} \text{ ie. } \frac{\text{K}}{\text{R}} = \frac{12}{\text{H cos z}} \quad \therefore \quad \text{K} = \frac{12\ \text{R}}{\text{H cos z}}.$$

It will be noted that fig. 44 pertains to any vertical plane.

Second half of Verse 40. The Dinārdha-Karṇa is equal to $\frac{\text{R} \times k}{\text{Hṛti}}$ where k is the Viṣuvat-Karṇa.

Comm. The formula is derived through twice applying the rule of three or what is the same, through the similarities of two sets of triangles

$$\text{From fig. 39, } \frac{\text{O}_1\text{A}}{\text{A}a} = \frac{\text{Hṛti}}{\text{Dinārdha-S'anku}} = \frac{k}{12} \quad \text{(a)}$$

where K is the required Chayākarṇa.

Dividing (a) by (b)

$$\overline{12} - 12 \ \hat{}\ \text{K} \quad \cdot\cdot \quad - \ \overline{\text{Hṛti}}$$

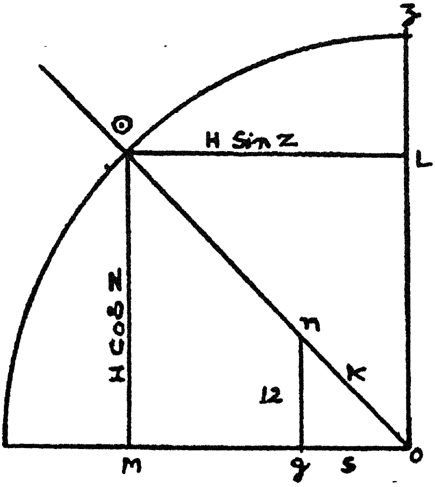


Fig. 44

Verse 41. Alternate method of obtaining K

101530/H sin λ = para (say) where λ is the Sāyana longitude of the Sun ; then, $\frac{\text{Para} \times k}{s}$ = K where K is un-mandala-Karṇa

First half of Verse 42. To obtain K when the Sun is on the prime-vertical— Para $\times$ s/k = Samavṛttakarṇa.

Comm. From fig. 19, from the similarity of triangles

CA

H sin δ $\quad \frac{\text{H sin } \lambda \text{ H sin } \omega}{\text{R}}$ (a)

Then consider the similarity of the first and the sixth latitudinal triangles ; then $\frac{\text{Unmandala S'anku}}{\text{Krāntijyā}} = \frac{s}{k}$ (b)

where s is the equinoctial shadow and k the Viṣuvat-Karṇa. Again taking that ⊙ the Sun lies on the unman-

dala in figure 44, $\frac{\text{H cos z}}{12} = \frac{\text{R}}{\text{K}} = \frac{\text{Unmandala S'anku}}{12}$ (c)

Eliminating Krāntijyā and Unmandala S'anku from (a), (b) and (c) $\frac{\text{Unmandala S'anku}}{\text{H sin } \lambda \text{ H sin } \omega/\text{R}} = \frac{s}{k}$

$12\text{R} / \frac{\text{H sin } \lambda \text{ H sin } \omega}{} = \frac{}{}$ ie. $\frac{}{\text{KH sin } \lambda \text{ H sin } \omega}$

$\frac{12\text{R}^2 \times k}{\quad \text{n } \omega}$. Here $\frac{12\text{R}^2}{\text{H sin } \omega} = \frac{12 \times 3438^2}{1397}$

$= 101531$; but Bhāskara has taken 101530 taking a more correct value of R. Then $\frac{101530}{\text{H sin } \lambda}$ is symbolized as para so that para $\times \frac{k}{s}$ = K = Unmandala Karṇa. Regarding the Samavṛttakarṇa, in the place of (b) above we have $\frac{\text{Sama-S'anku}}{\text{Krāntijyā}} = \frac{k}{s}$ (b′) by the similarity between the first and the fifth latitudinal triangles. Equation (c) holds good with respect to any H cos z and the corresponding K since 12 R = K × S'anku and 12 R is a constant. Noting therefore $\frac{\text{R}}{\text{K}'} = \frac{\text{Sama-S'anku}}{12}$ (c′)

eliminating Krāntijyā and Sama-S'ankn among (a), (b′), (c′), we shall have

$$\text{K} = \frac{12 \text{ R}^2 s}{k \text{ H sin } \lambda \text{ H sin } \omega} = \text{para} \times \frac{s}{k} \text{ as stated.}$$

Second half of Verse 42. To obtain the Dinārdhakarṇa from the Unmandalakarṇa.

$$\frac{\qquad}{\text{Antyā}} \; \frac{\text{Charajyā}}{\qquad} = \text{Dinārdhakarṇa.}$$

Comm. We have equation (c) above stating
12 R = K × S'anku. (c) But

$$\frac{\text{Iṣṭa S'anku}}{\text{Iṣṭa Hriti}} = \cos\varphi = \text{constant} = \frac{\text{Dinārdha S'anku}}{\text{Hṛti}}$$

$$= \frac{\text{Sama-S'anku}}{\text{Taddhṛti}} = \frac{\text{Unmandala S'anku}}{\text{Kujyā}}$$

Again by virtue of the proportionality of

$$\frac{\text{Iṣṭa Hṛti}}{\text{Iṣṭāntya}} = \frac{\text{Hṛti}}{\text{Antyā}} = \frac{\text{Kujyā}}{\text{Charajyā}} \quad (e)$$

We nave $\frac{\text{Ishta-S'anku}}{\text{Ishtāntyā}} = \frac{\text{Dinārdha S'anku}}{\text{Antyā}}$

$$= \frac{\text{S'anku}}{\text{Charajyā}}$$

∴ Iṣṭa Karṇa × Iṣṭāntya = Dinārdha Karṇa × Antyā
= Unmandala Karṇa × Charajyā (f) (25)

$$\therefore \text{Dinārdha Karṇa} = \frac{\text{Unmandala Karṇa} \times \text{Charajyā}}{\text{Antyā}}$$

as stated in the verse.

Verse 43. $\frac{\text{Unmandala Karṇa} \times \text{Kṣitijyā}}{\text{Hṛti}}$

$$= \frac{\text{Sama Vṛtta Karṇa} \times \text{Ta}}{\text{Hṛti}}$$

Comm. From (c) and (d) above Dinārdha Karṇa × Hṛti = Sama Karṇa × Taddhṛti = Unmandala Karṇa × Kujyā (g). Khitijyā is the same as Kujyā.

From this the statement follows:

Verse 44. The ancient Achāryas found the gnomonic shadows when the Sun is on the meridian, prime-vertical and the Kona-Vṛtta (ie, Vertical when the northern or southern Hindu azimuths are 45°) by different methods. I consider him to be the very Sun illuminating the lotus-faces of aitronomers, if anybody could give a method to find the shadow in any required direction, which holds good in all cases universally.

Comm. Evident.

Verse 45. Definition of Dikjyā H sin (azimuth). The angle between any vertical and the Prime-Vertical measured on the horizon is what is called Digamsa and its H sine is known as Dik-jyā either in the Eastern hemisphere or the Western.

Comm. In modern astronomy azimuth is measured along the horizon from the north point towards the east point round the horizon. In Hindu Astronomy however, the azimuth is measured from the East point on either side and from the West point also on either side specifying whether it is north or south.

Verse 46 *and first half of* 47. To obtain the gnomonic shadow in any arbitrary direction.

Assume $\frac{Rs}{\text{H sin } a}$ as the equinoctial shadow and obtain the H sine of the corresponding latitude L. Then the product of that H sin L and H sin δ divided by H sin φ will give H sin D where D is a hypothetical declination. With the new L and this D, as the hypothetical latitude and declination, obtain the meridian zenith distance by the formula $Z + D = \phi$, and through this *m.z d.* obtain the shadow, which will be the shadow in the required direction namely 12 tan $(\phi \mathbin{\overset{+}{\sim}} D)$.

Comm. Let gL be the gnomonic shadow on the equinoctial day in a given direction given by $a°$ Digamsa (the Hindu azimuth) and let gN be the shadow in the same direction on any day. (fig. 45) We know that the extremity of the gnomonic shadow on the equinoctial day traces a straight line parallel to the East-West line Eω at a distance of the equinoctial shadow s because the Equatorial plane passing through the foot of the gnomon

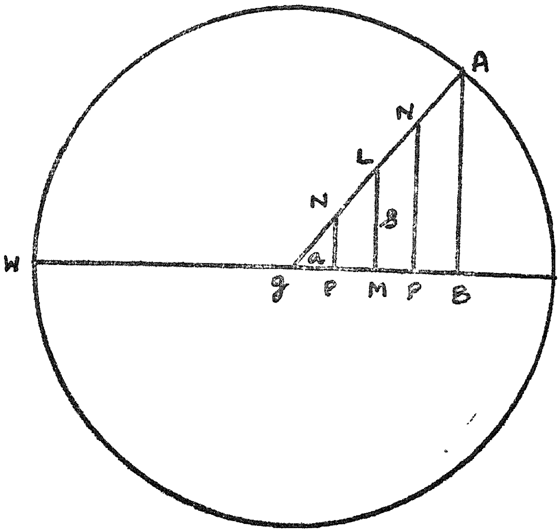


Fig. 45

and that passing through the top of the gnomon being parallel planes cut the horizontal plane in parallel straight lines. (This will be also proved analytically subsequently).

Hence LM $= s$. Now from the figure

$$\frac{LM}{gL} = \frac{AB}{gA} = H \sin a \qquad gL \qquad \frac{H \sin a}{Rs} \qquad I$$

This gL is spoken of as Iṣṭa-Drikmandala palabhā because it is the shadow on the equinoctial day in any vertical. LN is the increment in the shadow on account of declination and we have to compute this and correlate gL and LN. For this refer to figs. 46 and 47. In fig. 46, QRT is the equator, so that when the Sun is on the equator on the equinoctial day in the direction given by ZS, ZT is the zenith-distance. Let ZS be the zenith-distance of the Sun in the same direction on any day. From the analogy of finding H sin δ from H sin λ, from this figure

$$H \sin SR = \frac{H \sin ST \times H \sin T}{R} \quad \text{II and}$$

$$H \sin \varphi = \frac{H \sin ZT \times H \sin \hat{T}}{R} \quad \text{III so that}$$

$$\frac{H \sin SR}{H \sin \varphi} = \frac{H \sin ST}{H \sin ZT} \therefore H \sin ST = \frac{H \sin SR}{H \sin \varphi}$$

Noting that $SR = \delta$ and putting $ST = D$

$$H \sin D = \frac{H \sin \delta}{H \sin \phi} \times H \sin ZT.$$

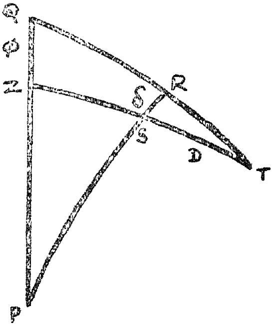


Fig. 46

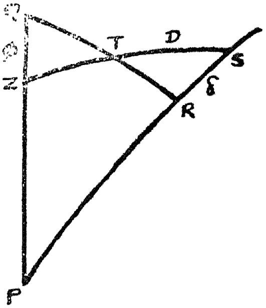


Fig. 47

The same formulae are derivable from fig. 47 also; only in fig. 46 while there is a decrement in the shadow of the day as compared with the shadow on the equinoctial day, in fig. 47, there is an increment. This is seen from the decrease and increase of ST in the zenith-distance ZT of the equinoctial day in the given direction. Now correlating fig. 45 with figures 46 and 47, the shadow gL pertains to the zenith-distance ZT on the equinoctial day whereas the shadows gN pertains to the zenith-distance on the day concerned in the same direction. We have,

$S \qquad H \sin z$ so that $\frac{RS}{\sqrt{12^2 + S^2}} = H \sin z$ where

S is the shadow at any instant when the zenith-distance is z. The process indicated by saying 'Obtain H sin φ

construing $\frac{Rs}{\text{H sin } a}$ as the equinoctial shadow', means computing H sin ZT and there from ZT from the shadow gL of fig. 45. Then the process indicated by saying "Obtain $\text{H sin D} = \frac{\text{H sin ZT} \times \text{H sin } \delta}{\text{H sin } \varphi}$ and therefrom D" means computing ST.

Then clearly $ZS = ZT \pm ST$, ie. the required zenith-distance is got by what is technically called Samskāra between ZT and ST as is stipulated between φ and δ to obtain Z from the formula $Z + \delta = \varphi$ (The word Samskāra was defined as meaning addition when the directions are the same and difference when they are opposite). Then the gnomonic shadow is got from this zenith-distance using the formula $S = \frac{12 \text{ H sin z}}{\text{H cos z}}$.

Thus the procedure adopted by Bhāskara was conceived by him first having Fig. 45 before him and then using figures 46 and 47. In this particular process, H sin a is given and H sin δ also, which means that it is sought to find the shadow on a given day in a given direction.

Incidentally we shall find the locus of the extremity of the gnomonic shadow during the course of a day. Let in fig. 48 g represent the gnomon's foot, and S the shadow whose extremity is p. Required to find the locus of p. Take the gnomon to be of unit length so that the length of the shadow $S = 12 \tan z$ becomes $\tan z$ here. Take Eω and sn the east-west line and the north-south as the axes. Then we have

$x^2 + y^2 = \tan^2 z$ (1). But we have from the triangle PZS $\sin \delta = \sin \varphi \cos z + \cos \varphi \sin z \sin a$

ie. $\frac{\sin \delta}{\cos \varphi \cos z} = \tan \varphi + \tan z \sin a = \tan \varphi + y \quad (2)$

ie. $\frac{\sec z}{A} = y + \tan \varphi$ when $A = \frac{}{\sin}$

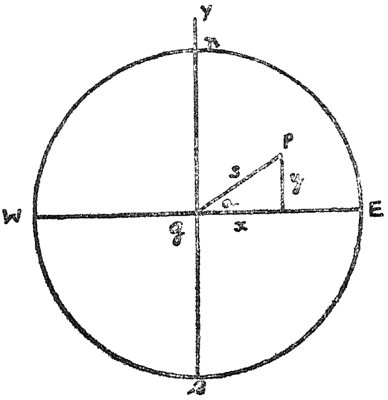


Fig. 48

But sec z $= \sqrt{1 + \tan^2 z} = \sqrt{x^2 + y^2 + 1}$

$\therefore \quad \dfrac{\sqrt{x^2 + y^2 + 1}}{A} = y + \tan \varphi$ which reduces to

$x^2 + y^2 (1 - A^2) - 2A^2 y \tan \varphi + 1 - A^2 \tan^2 \varphi = o$ (3)
From this it is evident that the locus is an ellipse or parabola or hyperbola according as $A \lesseqgtr 1$; also it will be seen that the eccentricity is A. The locus is wrongly stated to be always a hyperbola in some text books. For it to be an ellipse $A < 1$ ie. $\cos \varphi < \sin \delta$ ie. $\delta > 90 - \varphi$ ie. $\varphi + \delta > 90$. In such latitudes and under such declinations, it will be an ellipse ie. at a place just north of the place where the perpetual day just begins the locus will be an ellipse. Hence in the arctic region it will be always an ellipse; and in the place just at which the perpetual day begins it will be a parabola and in the lower latitudes it will be a hyperbola, ie. it will be a parabola where the latitude φ is given by $90 - \delta$.

When $\varphi = 90°$, A = $\qquad = o$ provided $\delta \neq o$. If, however, in addition $\delta = o$, A becomes indeterminate, but we may note then, that the Sun will be circling round the horizon on that equinoctial day at the north pole. We

may further note that at the north pole, the altitude of the Sun is always δ so that the length of the shadow cast is always equal to cot δ and this will be infinite when $\delta = o$.

If now $\phi = 90°$, and $\delta \neq o$, though $A = \frac{\cos \phi}{\sin \delta}$ becomes zero, A tan ϕ will not be zero because $A \tan \phi = \frac{\cos \phi}{\sin \delta} \times \tan \phi = \frac{\sin \varphi}{\sin \delta} = \frac{1}{\sin \delta}$ $(\because \varphi = 90°)$. On the other hand $A^2 \tan \varphi = A \times A \tan \varphi = \frac{1}{\sin \delta} = O$ because $A = o$. Thus the term containing y in eqn. (3) vanishes.

$\therefore$ The equation reduces to
$x^2 + y^2 = A^2 \tan^2 \phi - 1 = \text{cosec}^2 \delta - 1 = \cot^2 \delta$ (ie. +ve) ie. at the north pole the locus will be a circle with radius cot δ. When

$A = \infty$ the locus $\frac{\sqrt{x^2 + y^2 + 1}}{A} = y + \tan \phi$ becomes $y = - \tan \phi$ which means that the hyperbola degenerates into the straight line which is parallel to the east-west line and is in the north at a distance of tan ϕ ie. s, the equinoctial shadow since the length of the gnomon is taken to be unity. In particular when A tan $\phi = 1$ ie. $\varphi = \delta$, the constant in (3) is zero, so that the locus passes through the foot of the gnomon as is also evident from the fact that the Sun passes through the zenith.

Taking a northern latitude say 17°, the loci of the extremity of the shadow are shown in fig. 48A (page 270) on important days when $\delta = \omega$, when $\delta = o$, when $\delta = - \omega$, when $\delta = \varphi$. The maximum mid-day shadow is $\tan(\varphi + \omega)$, when $\delta = - \omega$ taking the gnomon's length to be unity; this shadow is cast north of the gnomon along the south-north line through the gnomon, on Dec. 23rd of the year. The minimum length of the mid-day shadow occurs when $\delta = \varphi$, the shadow being zero and being at the foot of the gnomon, the Sun being then just overhead. The maximum shadow cast south of the gnomon at mid-day is $\tan(\omega - \varphi)$.

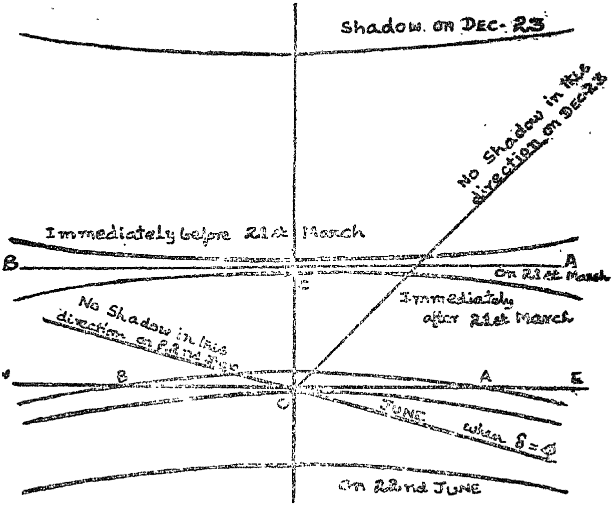


Fig. 48A Showing the locus of the extremity of the shadow on different days at a latitude of 17°

Note. OA, OB are the shadows computed by Bhāskara when the Sun is on the prime-vertical.

In the method of finding the shadow under verse 46, we perceive Bhāskara's genius in (1) looking upon the shadow as being made up of two segments namely that due to φ and that due to the declination (2) in conceiving what he calls Iṣṭa-drik-mandala palabhā, and Iṣṭa-drik-mandala Krānti and (3) in deriving the equation

$$\text{H sin D} = \frac{\text{H sin ZT} \times \text{H sin } \delta}{\text{H sin } \varphi}$$

Latter half of verse 47 *and verse* 48. Something to be noted.

In computing the shadow in a given direction, there may be two shadows at times in the northern hemisphere. When H sin a < Agrā and there will be none in the southern. To compute the second shadow we have to take 180 − L also as the latitude where L is the latitude computed, and proceed in the same way as we have done before.

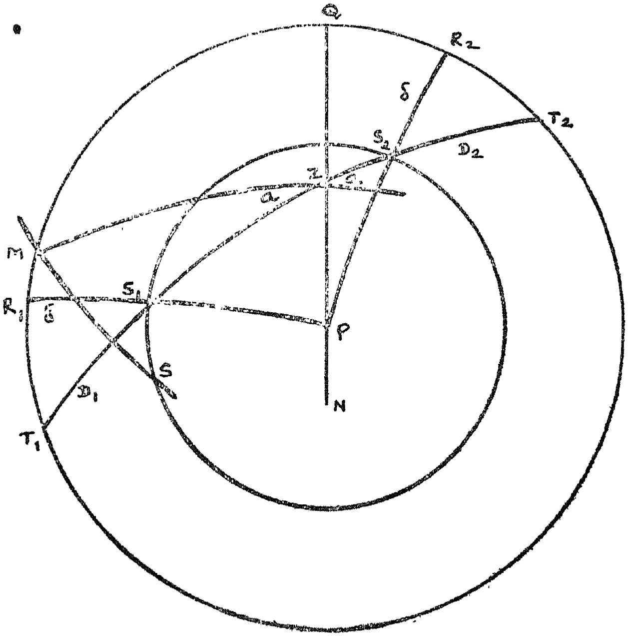


Fig. 49

Comm. This too exhibits Bhāskara's genius. (Ref. fig. 49). Let MQR_2 be the equator whose pole is *p*. Let $T_1\ S_1\ Z\ S_2\ T_2$ be the circle of azimuth a (Hindu azimuth). Let $SS_1\ S_2$ be the diurnal circle of the Sun cutting the above circle of azimuth at S_1 and S_2, so that ZS_1 and ZS_2 are the two solutions giving the two zenith-distances which give two shadows in the given direction. H sin MS = Agrā ; evidently $\widehat{MZS} > \widehat{MZS_1}$ ie. H sin $a <$ Agrā as stipulated. ZT_1 and ZT_2 give the zenith-distances in the given direction when the Sun is on the equator. $S_1\ T_1$ and $S_2\ T_2$ are the decrements in the zenith-distances on account of declination δ (= $S_2\ R_2$ or S_1 If MZS_1 were greater than $\widehat{MZS}$ ie. if H sin $a >$ Agrā, we would have lost the position S_1 ie. we would have had only one shadow

in the afternoon in the given direction and no shadow in the morning.

Analytically, the event of having two shadows arises on account of the following circumstance. When we are asked to find Iṣṭākṣajyā from $\frac{Rs}{H \sin a}$ the shadow in the given direction on the equinoctial day (Ref. gL fig. 45) the L for this given value of the shadow is given by

H L = where S = $\frac{Rs}{\cdot}$. We know

sin $\theta = a$ has two solutions, θ_1 and 180 − θ_1. Hence L will have two values L_1 and 180 − L_1. So, Bhāskara has asked us to compute D_1 and D_2 from L_1 and L_2 and thus have the two solutions.

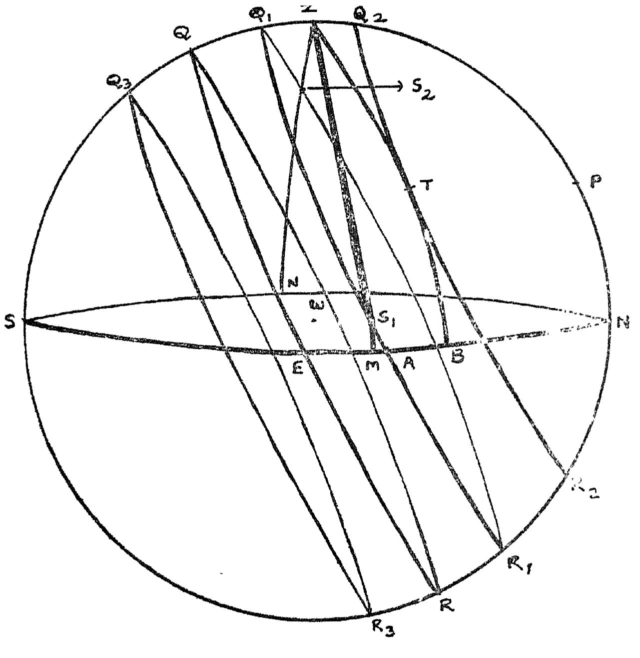


Fig. 50

Note (1) When Bhāskara said 'If H sin $a <$ Agrā' he had in mind evidently the azimuth circle MZN which cuts the diurnal path A $Q_1 R_1$ at S_1 and S_2. At S_1 the azimuth EZM $<$ EZA so that he stipulated that H sin a should be less than the Agrā. But, let the diurnal path of the Sun be $Q_2 R_2$ where Q_2 falls in between z and p. In such a case we know that the azimuth does not take all values but has a maximum where the vertical touches the diurnal path at T. From PTZ where T is a right angle, taking PZT $= 90 - a$, a being the Hindu azimuth we have by Napier's rule sin PT $=$ sin ZP sin $(90 - a)$ or

$\sin(90-\delta) = \sin(90-\varphi)\sin\overline{90-a}$ or $\cos\delta = \cos\varphi\cos a$. If a has a lesser value than is given by this equation, the diurnal path does not cut the azimuth circle ie. if $\cos a > \dfrac{\cos\delta}{\cos\varphi}$, there will be no shadow in the given direction even though the situation satisfies Bhāskara's condition namely H sin a should be less than Agrā. *Bhāskara has overlooked this case.* This may be seen analytically also as follows. We have from the spherical triangle PZS, $\sin\delta = \sin\varphi\cos z + \cos\varphi\sin z\sin a$ $= A\cos z + B\sin z$ (say). We know, the maximum value of $A\cos z + B\sin z$ is $\sqrt{A^2 + B^2}$ which is here $\sqrt{\sin^2\varphi + \cos^2\varphi\sin^2 a} = \sqrt{\sin^2\varphi + \cos^2\varphi - \cos^2\varphi\cos^2 a}$ $= \sqrt{1 - \cos^2\varphi\cos^2 a}$. Thus there will be no solution for z if the quantity on the left hand side namely

$\sin\delta >$ the above max. value ie. if

$\sin\delta > \sqrt{1 - \cos^2\varphi\cos^2 a}$ ie. if $\sin^2\delta > 1 - \cos^2\varphi\cos^2 a$

ie. if $\cos^2\delta > \cos^2\varphi\cos^2 a$ ie. if $\cos\delta > \cos\varphi\cos a$

ie. if $\cos a > \cos\delta/\cos\varphi$ as derived above.

Hence even if H sin $a >$ Agrā, there need not be a shadow at all in the given direction. In other words when the Hindu azimuth given is very small and when the declination is too great north or south, there may not be a

shadow in the given direction. Bhāskara gives an example where he gets two shadows on a day taking the moments when the Sun is on the prime-vertical. In fact having this case of the East-West shadows alone, he conceived that two shadows could be had in a given direction under particular conditions.

He chooses a place of $s = 5''$ ie. a place of latitude $22° - 37'$ (Bhāskara often gives this latitude which might indicate that he was probably residing in that latitude which passes through approximately Itarsi). He takes a day when the Sun's declination is given by H sin $\delta = 780$ ie. $\delta = 13° - 7'$. Then the Sama-S'anku is given by $\frac{R \sin \delta}{\sin \varphi}$ (comparing the second and the fifth latitudinal triangles

$$\frac{\text{Sama-S'anku}}{R} = \frac{\text{Krantijyā}}{H \sin \varphi} = \frac{H \sin \delta}{H \sin \varphi}$$

$$\therefore \text{ Sama-S'anku} = \frac{RH \sin \delta}{H \sin \varphi} = \frac{I}{\sin \varphi}$$

$$= \frac{3438 \times 780}{} = 2028 \text{ approximately.} \quad I$$

$$\text{Also Agrā} = \frac{R \sin \delta}{\cos \varphi} = 845.$$

Knowing the Sama-S'anku, the East-West shadow may be taken to be determined. Now Bhāskara proceeds to show that at the time of having the second shadow also, in the same East-West direction, the S'anku will be the same Sama-S'anku itself. For this, proceeding according to the method indicated in the verse, " Taking $\frac{Rs}{H \sin a}$ to be the equinoctial shadow etc. " we have

$$\frac{Rs}{H \sin a} = \frac{3438 \times 5}{0} = \text{what is called Kha-hara Rās'i.}$$

Taking this to be the equinoctial shadow

H sin L = $\frac{Rs}{\sqrt{12^2+s^2}}$ = R itself (Dealing this way with Kha-hara Rāśis is prohibited in modern mathematics but Bhāskara adds at the end of the commentary that dealing with them cautiously does not effect computations which is of course true, for when the equinoctial shadow is infinity $\varphi = 90°$ so that H sin φ = R as got). Hence L = 90° and 180° − 90° = 90° = L′. Then H sin D = $\frac{R \times 780}{1322 - 18}$ = same as Sama-S'anku obtained in I = H cos z so that D = 90 − z. Now from the equation z + $\delta = \varphi$, z = $\varphi - \delta$ = L′ − D = 90° − (90 − z) where z is the zenith-distance when the Sun is on the prime-vertical.

∴ The zenith-distance is again the same z. In other words, the second zenith-distance is also that when the Sun is on the prime-vertical. Bhāskara has given this example just to obtain the second shadow as well and he has chosen the event of the Sun being on the prime-vertical to show that the procedure indicated by him may be verified to hold good.

Verses 49, 50. Alternate method to find the shadow.

Let $R^2 s^2$ + H $\sin^2 a \times 12^2$ = prathama where s = equinoctial shadow and a the Hindu azimuth. Let Anya = RsA where A is the Agrā. Divide the prathama and Anya by (H $\sin^2 a - A^2$) and still call them prathama and Anya. Then

K = $\sqrt{\text{Adya} + \text{Anya}^2} \pm$ Anya where K is the Chāyā-

Comm. Let K be the required Chayā-Karṇa.

Then Karnāgrā = $\frac{KA}{R}$ = $s + b$ where b is the bhuja.

$= \frac{KA}{R} - s$. But $\frac{H \sin a}{b} = \frac{R}{S}$ where S is the shadow so that

$$S = \frac{bR}{H \sin a} = \left(\frac{KA}{R} - s\right) \frac{}{H \sin a}$$

$$\therefore R(KA - sR) = RH \sin a \sqrt{K^2 - 12^2}$$

ie. $R^2 (K^2 A^2 + s^2 R^2 - 2 AsRK = H \sin^2 a (K^2 - 12^2)$

$$\therefore K^2 (A^2 - H \sin^2 a) - 2 AsRK = - s^2 R^2 - 12^2 H \sin^2 a$$

$$\therefore K^2 (H \sin^2 a - A^2) + 2 AsRK = 12^2 H \sin^2 a + s^2 R^2 \quad I$$

This quadratic is of the form $ax^2 + 2bx = c$

ie. $x^2 + 2\frac{b}{a}x = c/a$ II Here $\frac{'c'}{a}$ is called Adya and $\frac{b}{a}$ Anya.

The solution of the above equation is given by $x = -\frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} + c/a}$

ie. $-$ Anya $\pm \sqrt{\text{Anya}^2 + \text{Adya}}$. III

When $b = \frac{KA}{R} + s$, putting $-s$ in the place of s in I,

K = Anya $\pm \sqrt{\text{Anya}^2 + \text{Adya}}$ IV

Out of the four solutions given by III and IV we have taking the positive solutions

K = $\sqrt{\text{Anya}^2 + \text{Adya}} \pm$ Anya as stated.

Verse 51. If H sin $a <$ A, then in the northern hemisphere ie. where δ is north,

$\pm \sqrt{\text{Anya}^2 - \text{Adya}}$ + Anya = K.

Comm. We have initially put Anya = H $\sin^2 a - A^2$. If H sin $a <$ A, them to avoid a negative value for the

Anya, we could put Anya $= A^2 - H\sin^2 a$. As a matter of fact in verse 50, we are asked to take $H\sin^2 \sim A^2$, as Bhāskara wanted that the second case also be included.

Thus putting Anya$=A^2-H\sin^2 a$, equation I becomes $K^2(A^2 - H\sin^2 a) - 2\,AsRK = -$ (Adya) so that
$K =$ Anya $\pm \sqrt{\text{Anya}^2 - \text{Adya}}$ as given.

Verse 52. The Bhuja is to be obtained through Karṇāgra from the equation $a = b + s$ (already proved) and $\frac{Rb}{S} = H\sin a$ {ie. $S = \frac{bR}{H\sin a}$ as already proved}.

This H sin a will be the same in the case of obtaining two values of K ie. two shadows one in the morning and the other in the afternoon, (the only difference being that they will be on alternate sides of the East-West line).

Verses 53 *and* 54. Obtaining the shadow when time is given.

In the two previous examples the magnitude of the shadow was obtained in a given direction; now we shall obtain the same when time is given. The word unnata stands for the time that has elapsed after Sun-rise or that which is the balance of the day time. The unnata subtracted from half-the-day gives what is called Nata.

The H sine of the unnata minus Chara or increased by the Chara according as the Sun's declination is north or south, is called Sūtra; this multiplied by the H cos δ and divided by the radius, is called Kalā.

Comm. (Ref fig. 51) The time measured by the arc MN, that is the time in between the moment when the Sun S is on the horizon and the moment when he is at L is called the unnata ie. the time measured after rising and the time measured by the arc NQ ie. the time in

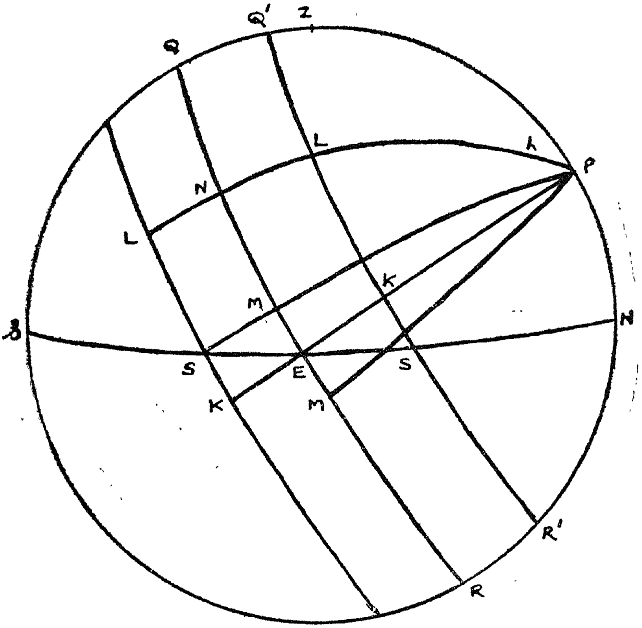


Fig. 51

the moments when the Sun is at L and when he reaches the meridian. is called Nata. In modern astronomy this Nata is known as the hour angle h and unnata $= H - h$ where H is the rising hour angle. The time measured by the arc ME is called Chara. Thus unnata-chara = EN and H sin EN H sin $(90 - h)$ = H cos h is called Sutra which is BN shown in fig. 52 representing the Equator. The corresponding line bn in fig. 53 which represents the diurnal circle, is known as Kalā. Thus Sūtra = H cos h (26) and Kalā $= \dfrac{H \cos h \times H \cos \delta}{R}$ (27).

When the Sun is in the Southern hemisphere, unnata is measured by the arc SL or MN ie. $(H - h)$ where H is

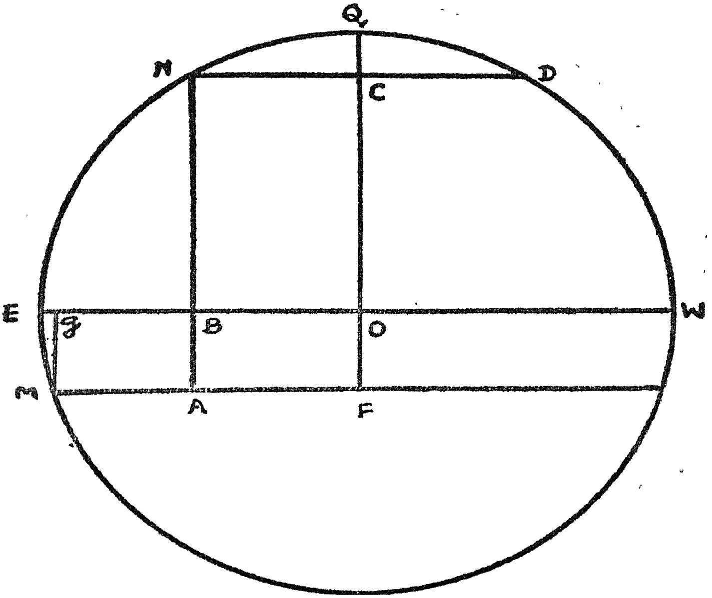


Fig. 52

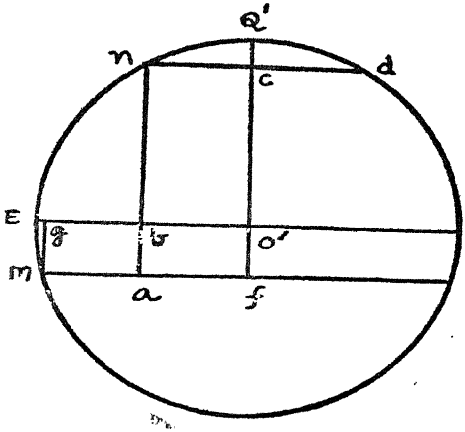


Fig. 53

the rising hour angle and $h = \hat{LPQ}$ and Chara by the arc EM and Sūtra = H sin (MN + EM) = H sin (unnata + chara) = H sin EN = H sin (90 − h) = H cos h.

Verse 55. Sūtra multiplied by Kujyā and divided by Charajyā will be also Kalā and Kalā multiplied by the Koti and divided by the Karṇa of any latitudinal triangle will be Iṣta *yaṣti*.

Comm. In as much as Kalā is the corresponding line in the diurnal circle to the Sūtra in the plane of the celestial equator

$$\frac{\text{Sūtra}}{\text{Kalā}} = \frac{R}{H\cos\delta} = \frac{\text{Charajyā}}{\text{Kujyā}} \quad \therefore \frac{\text{Sūtra} \times \text{Kujyā}}{\text{Charajyā}} = \text{Kalā}$$

In verse 33, we saw that

Dinārdha S'anku − Unmandala S'anku = Yaṣti.

This Iṣta-yaṣti will be therefore the perpendicular dropped from n, the Sun's position in the diurnal circle on the plane parallel to the horizon and passing through the head of the Unmandala S'anku ie. passing through B and parallel to the horizon in fig. 21. Since the angle between the diurnal plane and the vertical plane of the yaṣti is equal to φ the latitude, the Iṣtayaṣti forms a latitudinal triangle with the Kalā, it being the Koti or side opposite to the angle $90 - \varphi$ and the Kalā being the hypoten use,

$$\therefore \frac{\text{Yaṣti}}{\text{Kalā}} = \cos\varphi = \frac{\text{Koti of a latitudinal triangle}}{\text{Karṇa of a latitudinal triangle}}$$

$$\therefore \text{Iṣtayaṣti} = \text{Kalā} \times \frac{\text{Koti of a latitudinal triangle}}{\text{Karṇa}}.$$

When the Sun is on the meridian, Iṣtayaṣti becomes yaṣti of verse 33.

The formula for Iṣtayaṣti is therefore

$$\frac{\text{Kalā} \times H\cos\varphi}{R} = \frac{H\cos\delta \; H\cos h}{R} \times \frac{H\cos\varphi}{R}$$

from formula (27)

$$= \frac{H\cos\varphi \; H\cos\delta \; H\cos h}{R^2} \qquad 27'$$

First half of verse 56. The Unmandala S'anku multiplied by the Sūtra and divided by Charajyā is also Iṣṭayaṣṭi.

Comm. The Unmandala S'anku and Iṣṭayaṣṭi are the lines in vertical planes corresponding to Charajyā and Sūtra in the Equatorial plane. Hence the proportion. It will be noted that the Unmandala S'anku and Iṣṭayaṣṭi are not in the same vertical plane but parallel vertical planes. None the less the proportionality holds good.

Latter half of verse 56 and first half of verse 57.

The Sūtra increased or decreased by the Charajyā according as the Sun is in the northern or southern hemisphere is what is known as Iṣṭāntyā; similarly the Kalā increased or decreased by Kujyā is what is known as Iṣṭa-Hṛti.

Comm. In fig. 52, Iṣṭāntyā = AN = AB + BN = ME + BN = Charajyā + Sūtra. Similarly in fig. 53, Iṣṭa Hṛti = $an = ab + bn = sg + bn$ = Kujyā + Kalā.

$\therefore$ Iṣṭāntyā $= H \cos h + R \tan \delta \tan \varphi =$

$R \dfrac{(\sin \varphi \sin \delta + \cos \varphi \cos \delta \cos h)}{\cos \varphi \cos \delta}$ in modern terms (28)

Latter half of verse 57. Similarly Iṣṭayaṣhṭi increased or decreased by the Unmandala S'anku is Iṣṭa S'anku or H cos z.

Comm. Let in fig. 54 which represents the plane of the prime-vertical AA′, EW, BB′, FF′, qq′ represent the lines of intersection of this plane with planes parallel to the horizon and passing through A, B, F, q of fig. 21. Then $O\alpha$ = Unmanda-S'anku, $O\beta$ = Sama S'anku, $O\gamma$ = Dinārdha S'anku. If xx' be the line of intersection of this plane of the prime-vertical with a plane passing through an arbitrary position of the Sun in the diurnal circle and

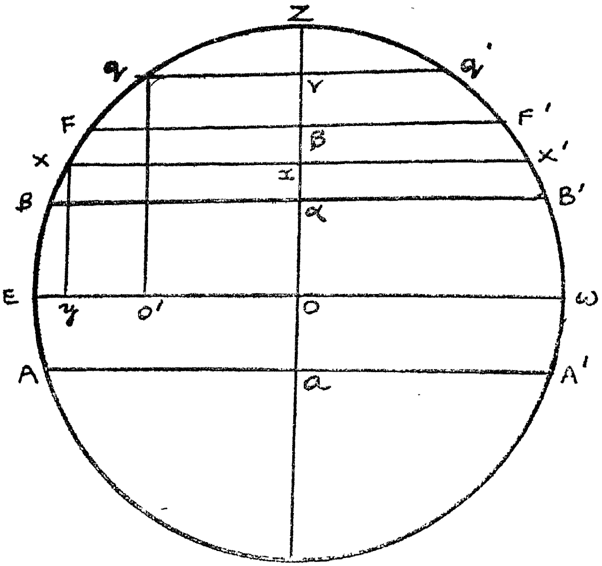


Fig. 54

parallel to the horizon. then Ox = Iṣṭa-S'anku = $O\alpha$ + αx = Un-mandala S'anku + Iṣṭa yasti. αr = yasti.

In the Southern hemisphere $O\alpha$, the Unmandala S'anku will be below the horizon so that Iṣṭayaṣṭi decreased by the Unmandala S'anku will be Iṣṭa-S'anku.

Thus we have the method of obtaining the Iṣhṭa-S'anku from the Unnata Kāla as detailed above. We shall see what this process means in modern terms.

Unnatakāla-Charakāla = $\odot \hat{P} A - A \hat{P} E$ (Fig. 21) = $\odot \hat{P} E$ where $\odot$ is the foot of the declination circle of the Sun e in any arbitrary position in his diurnal path. But $\odot PE = Q\hat{P}E - Q\hat{P}\odot = 90-h$ where h is the hour an

Sun, Thus Sutra = H sin $(90-h)$ = H

$$\therefore \quad \text{Kalā} = \frac{\text{H} \cos h \times \text{H} \cos \delta}{\text{R}} \quad \text{II}$$

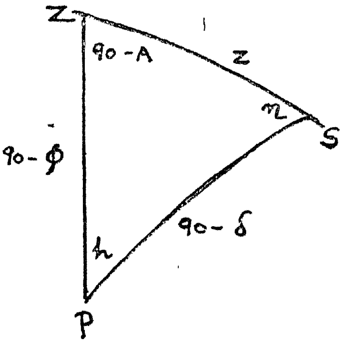
$$\therefore \quad \text{Iṣṭa-yaṣṭi} = \frac{\text{H} \cos h \times \text{H} \cos \delta}{\text{R}} \times \frac{\text{H} \cos \varphi}{\text{R}} \quad \text{III}$$

Now Unmandala-S'anku is derivable from the sixth latitudinal triangle in which Krāntijyā is the Karṇa and Unmandala-S'anku is the Bhuja. Comparing it with the second latitudinal triangle $\frac{\text{Krāntijyā}}{\text{R}} = \frac{\text{U. S.}}{\text{H} \sin \varphi}$ where U. S. is Unmandala-S'anku.

$$\therefore \quad \text{U. S.} = \frac{\text{Krāntijyā} \times \text{H} \sin \varphi}{\text{R}} = \frac{\text{H} \sin \varphi \ \text{H} \sin \delta}{\text{R}} \quad \text{IV}$$

∴ Iṣṭa-S'anku as per the above formulation is given by Iṣṭa-S'anku (I. S.) =

$$\frac{\text{H} \sin \delta \ \text{H} \sin \delta}{\text{R}} + \frac{\text{H} \cos \varphi \ \text{H} \cos \delta \ \text{H} \cos h}{\text{R}^2} = \text{I}$$



(Ref. fig. 55) Fig. 55 Formulae for

Z = zenith ; P = celestial pole. S = celestial body, say, the Sun.

= zenith-distance of the celestial body S.

PZ = colatitude; PS = north-polar-distance or co-declination.

n is called the parallactic angle; h = hour-angle of S. $(90 - a)$ = The complement of the Hindu azimuth a being measured from the East point.

$$\cos(90-\delta) = \cos(90-\varphi)\cos z + \sin(90-\varphi)\sin z \cos(90-a)$$

$$\} = \sin\varphi + \cos z + \cos\varphi \sin z \sin a \quad (1)$$

$$\cos z = \cos(90-\varphi)\cos(90-\delta) + \sin(90-\varphi) \times \sin(90-\delta)\cos h$$

$$= \sin\varphi \sin\delta + \cos\varphi \cos\delta \cos h \quad (2)$$

$$\cos(90-\varphi)\cos(90-a) = \sin(90-\varphi)\cot z - \sin(90-a)\cot h$$

ie. $\sin\varphi \sin a = \cos\varphi \cot z - \cos a \cot h$ (3)

$$\cos(90-\varphi)\cos h = \sin(90-\varphi)\cot(90-\delta) - \sin h \times \cot(90-a)$$

ie. $\sin\varphi \cos h = \cos\varphi \tan\delta - \sin h \tan a$ (4)

$$\frac{\sin}{\sin(90-a)} \qquad a = \sin h \cos\delta \quad (5)$$

This means in modern terms
$\cos z = \sin\varphi \sin\delta + \cos\varphi \cos\delta \cos h$ V which we derive from the triangle PZS.

58. To get H cos z from h the hour-angle or nata Kāla.

The H. vers (Nata) is called S'ara (CQ of fig. 52) (29)

Antyā − S'ara = Iṣṭāntyā ie. FQ − CQ = FC = AN (fig. 51)

(30)

Hṛti — phala = Iṣṭa-Hṛti ie. $fq - cq = fc = an$ (fig. 53)

Comm. (Ref. fig. 52). Nata = arc QN

∴ H. vers (Nata) = QC. (Called S'ara) = H. vers (h)

Antyā — S'ara = FQ – CQ = FC = AN = Iṣṭāntya II

$$\text{S'ara} \times \frac{H \cos \delta}{R} = \text{phala} = cq \text{ (fig. 53)} = \frac{H\text{vers } h \times H \cos \delta}{R}$$

Hṛti — phala = $fq - cq = fc = an$ = Iṣṭa-hṛti IV

§ 59. $\dfrac{\text{Phala} \times \text{Koti of a latitudinal triangle}}{\text{Karṇa}}$

= Ūrdhwa V

= βr (of fig. 54)

Comm. (Ref. fig. 54) βr is the corresponding line in the plane of the meridian corresponding to phala in the plane of the diurnal circle. As the angle between these two planes is the latitude itself, by the principle of orthogonal projection namely.

Magnitude of a projected segment = cosine of the dihedral angle × the magnitude of the segment projected, since Ūrdhwa is the orthogonally projected segment of phala, so, Ūrdhwa = phala × cos φ

= phala × $\dfrac{H \cos \varphi}{}$ VI

$$= \frac{\text{phala} \times \text{Koti of a latitudinal triangle}}{\text{Karna of the latitude triangle}} \text{ as stated}$$

Thus Ūrdhwa = βr (of fig. 54).

§ 60. Ūrdhwa is also given by

$$\text{Ūrdhwa} = \frac{\text{U.S. (Unmandala S'anku)} \times \text{S'ara}}{\text{Charajyā}}.$$

Dinārdha-S'anku (D.S.)—Ūrdhwa = Iṣṭa-S'anku (I.S.). H cos z.

Comm. Since U.S. is the projected segment of Charajyā on a vertical plane and since the diahedral angle between the planes is φ the latitude

$$\frac{\text{U.S.}}{\text{Charajyā}} = \text{Cos}\ \varphi \text{ and so S'ara} \times \cos\varphi = \frac{\text{S'ara} \times \text{H}\cos\varphi}{\text{R}}$$

= Ūrdhwa.

From fig. 54, Dinārdha-S'anku – Ūrdhwa = $o'q - \beta r$ (fig. 54) = $o'\beta' = y$X = Iṣta-S'anku.

In modern times, this means,

S'ara = H. vers (h) = (R – cos h) = 1 – cos h in modern terms

$$\text{phala} = \frac{\text{S'ara} \times \text{H}\cos\delta}{\text{R}} = (1-\cos h) \times \cos\delta \quad ,,$$

$$\text{phala} \times \frac{\text{H}\cos\varphi}{\text{R}} = \text{Ūrdhwa} = (1-\cos h)\cos\delta \times \cos\varphi \quad ,, \quad ,,$$

Dinārdha S'anku – Ūrdhwa = Iṣta S'anku H cos z
= cos z (in modern terms)

= H cos ($\varphi - \delta$) (taking northern declination and following Hindu convention with respect to signs)
= Dinārdha-S'anku
= cos $\varphi - \delta$ in modern terms.

$\therefore \cos(\varphi - \delta) - (1 - \cos h)\cos\varphi\cos\delta = \cos z$

ie. $\cos\varphi\cos\delta + \sin\varphi\sin\delta - \cos\varphi\cos\delta + \cos\varphi\cos\delta\cos h = \cos z$

ie. $\cos z = \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos h$ as before.

Verse 61. Computation of H cos z (Mahā S'anku) through Antyā and Hṛti.

Let the Dinārdha S'anku (D.S.) be computed through Iṣtāntyā and Iṣta Hṛti and therefrom Iṣta S'anku. From the S'anku, Drik-jyā ie. H sin z and the shadow $\frac{\text{KH}\sin z}{\text{R}}$ could be computed –*H sin z should not be computed* from Hṛti.

Comm. We computed Dinārdha S'anku by the formula.

$$D.S. = \frac{\text{Antyā} \times \text{U.S.}}{\text{Charajyā}} = \frac{\text{Hṛti} \times \text{Koti of a L.T.}}{\text{Karṇa of L.T.}}$$

under verse 36 ; similarly

Iṣta S'anku (I.S.) will be given by

$$\text{Iṣta S'anku} = \frac{\text{Iṣtāntya} \times \text{U.S.}}{\text{Charajyā}} = \frac{\text{Iṣta Hṛti} \times \text{Koti of a L.T.}}{\text{Karṇa of the L.T.}}$$

$$\text{But Iṣtāntyā} = \frac{R\ (\sin\varphi \sin\delta + \cos\varphi \cos\delta \cos h)}{\cos\varphi \cos\delta}$$

(as under verse 56)

$$\therefore \text{Iṣta S'anku} = \frac{R\ (\sin\varphi \sin\delta + \cos\varphi \cos\delta \cos}{\cos\varphi \cos\delta}$$

$\times \frac{R \sin\delta \sin\varphi}{R \tan\varphi \tan\delta}$ from formulae (13) and (19)

$= R\ (\sin\varphi \sin\delta + \cos\varphi \cos\delta \cos h) = R \cos z = H \cos z$

Or again Iṣta Hṛti $= \frac{R \cos z}{\cos\varphi}$ from formula (11)

$\therefore$ Iṣta S'anku $= \frac{R \cos z}{\cos\varphi} \times \frac{H \cos\varphi}{R} = R \cos z = H \cos z$

Having got H cos z, using the formula $H \sin^2 z = R^2 - H \cos^2 z$, H sin z ie. Dṛk-jyā can be computed. Also $K = \frac{12R}{H \cos z}$ and $S = \frac{KH \sin z}{R}$ give the Chāyā Karṇa and Chāyā. Bhāskara cautions us that H sin z cannot be computed from Hṛti as mentioned in verse 37. because there in that verse, the H sin z computed is that at noon alone.

Verse 62. Alternate method of obtaining K.

The Chāyākarṇa when the Sun is on the unmandala multiplied by Kujyā or that when the Sun on the prime-vertical multiplied by Taddhṛti, or again that when the Sun is on the meridian multiplied by Hrti, divided by Iṣta Hṛti, will be equal to the Iṣta-Karṇa K.

Comm. Equation (23) under verse 41 is

$$\frac{\text{Iṣṭa S'anku}}{\text{Iṣṭa Hṛti}} = \frac{\text{D.S.}}{\text{Hṛti}} = \frac{\text{S.S.}}{\text{Taddhṛti}} = \frac{\text{U.S.}}{\text{Kujyā}} = \cos \varphi \quad \text{I}$$

and $\frac{12}{K} = \frac{H \cos z}{R}$ (under verse 40)

$$\frac{12R}{\text{Iṣṭa S'anku}} \text{ which means}$$

Dinārdha Karṇa $= \frac{12R}{\text{D.S.}}$; Sama Karṇa $\frac{12R}{\text{S.S.}}$

and unmandala Karṇa $= \frac{12R}{\text{U.S.}}$.

Substituting for the numerators in I

$$\frac{12R}{K \times \text{Iṣṭa Hṛti}} = \frac{12R}{DK \times \text{Hṛti}} = \frac{12R}{S.K \times \text{Taddhṛti}}$$

$$= \frac{12R}{U.K. \times \text{Kujyā}} \quad \text{II}$$

$\therefore$ Iṣṭa Karṇa $\times$ Iṣṭa Hṛti $=$ *S.K.* $\times$ Taddhṛti

$=$ U.K. Kujyā

$$\therefore \text{Iṣṭa Karṇa} = \frac{\text{Uumandala Karṇa} \times \text{Kujyā}}{\text{Iṣṭa Hṛti}}$$

$$= \frac{\text{Sama Karṇa} \times \text{Taddhṛti}}{\text{Iṣṭa Hṛti}} = \frac{\text{Dinārdha Karṇa} \times \text{Hṛti}}{\text{Iṣṭa Hṛti}}$$

Verse 63. Just a caution.

If in any context where the word ūna-yuta has been used, the quantity to be subtracted exceeds the quantity from which it is to be subtracted, it goes without saying that subtraction should be reversely effected and in the place of addition subsequently prescribed subtraction should be done and Vice-versa.

Comm. Bhāskara gives three examples to illustrate his point. In verse 54, while defining Sūtra ($H \sin \overline{90 - h}$) we are asked to subtract chara from unnata when $\delta > o$

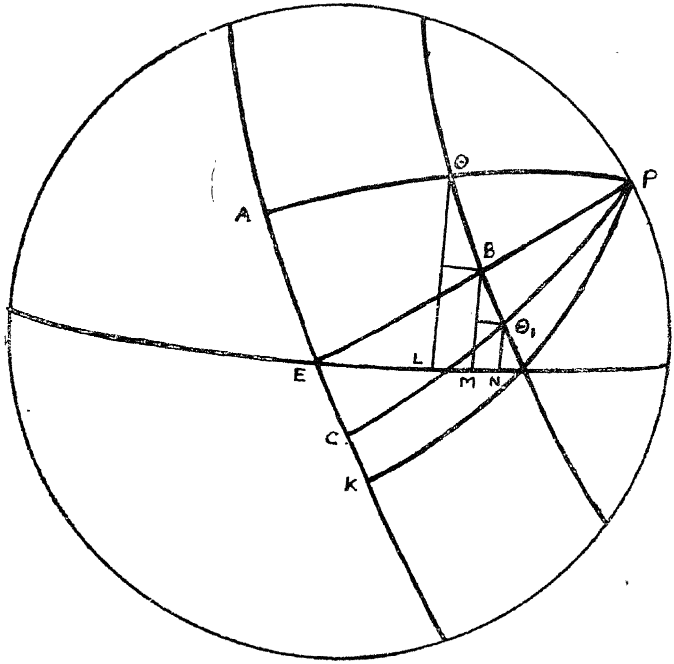


Fig. 56

and add Chara to Unnata when $\delta < o$ and take the H sine of the result. Let us first consider the case when $\delta > o$. (Refer fig. 56) when the Sun is at ⊙, Iṣta S'anku is H sin ⊙L; and Unmandala S'anku is H sin BM. When the Sun is at $\odot_1$, Iṣta S'anku = H sin $\odot_1$N. In the first case Iṣtayaṣti = (H sin ⊙ L — H sin BM) which will be the orthogonal projection of ⊙ B on the meridian plane. Unmandala S'anku and the Iṣta S'anku in the position $\odot_1$ are similarly the orthogonal projections of BM and $\odot_1$N on the same plane. In the position ⊙ Iṣta S'anku = Unmandala S'anku + Iṣtayaṣti, whereas in the position $\odot_1$, Iṣta S'anku = Unmandala S'anku — Iṣtayaṣti which is now downwards. Thus in the place of addition we have subtraction of Iṣtayaṣti. This reversion has arisen out of

the fact that in the position ⊙, Sūtra is the H sine of (KA — KE) whereas in the position $\odot_1$ Sūtra is the H sine of EC ie. H sine of (KE — KC) ie. in the former position Sūtra = H sine (Unnata-Chara) and in the lattter Sūtra = H sin (Chara-Unnata). Thus a reversion in subtraction here, effects a reversion of addition of the Iṣṭayaṣṭi.

Similar is the case in the other cases cited by Bhāskara. Analytically this happens so because cos h, when $h > 90$, becomes negative and adding cos h tantamounts to subtracting sin θ where $h = 90 + \theta$.

Verse 64. Another point to be observed.

Hvers $(90 + \theta)$ = R — H cos $\overline{90 + \theta}$ = R + H sin θ.

The Unmandala S'anku is not observable when δ is south in as much as it is below the horizon ; none the less it may be computed for the purposes of effecting proportion.

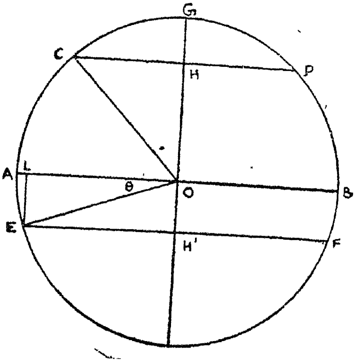


Fig. 57

Hvers (CG) = Hvers $(\widehat{COG})$ = GH

Hvers (EG) = Hvers $(\widehat{EOG})$ = Hvers $(90 + \theta)$

= R + OH^1 = R + EL = R + H sin ⊙

as defined by Bhāskara.

Comm. Under verse 58, we had to form Hvers h; a doubt might arise as to what this Hvers h would be if $h > 90$. Hence Bhāskara defines it and the definition is clear from fig. 57. The Unmandala S'anku has the formula R sin δ sin φ so that either when δ is negative or when φ is negative, it will be negative which means it will be in the opposite direction ie. vertically downwards. Since negative latitudes are not considered by the Hindu astronomers, the other case alone is considered. Even from a figure it is evident that when the Sun is in the south of the Equator, the Unmandala S'anku is vertically downwards. In proportions like I given under verse 62, we can use the magnitude of this Unmandala S'anku also and it does not vitiate the results, when we take its numerical value.

Verse 65. The Sun crosses the prime-vertical when his northern declination falls short of the latitude. Then alone there is sense in giving the magnitude of his shadow at that moment. When the Sun does not cross the prime-vertical at all, the Sama S'anku which could be computed out of its formulation, though it does not exist, under the Sun, yet, it could be used in proportions (like I under verse 62) and no blunder is committed.

Comm. This is a beautiful example cited by Bhāskara where he intuitively uses the so-called principle of geometrical continuity. We have formulated Sama S'anku as $\frac{R \sin \delta}{\sin \varphi}$ ie. $H \cos z = \frac{R \sin \delta}{\sin \varphi}$. We have a real value of z when $\delta > \varphi$, for, then only $H \cos z < R$. When $\varphi > \delta$, then $H \cos z > R$ which is impossible, for, no Hindu sine or Hindu cosine could be greater than R just as no modern sine or cosine could be greater than unity.

TERMINOLOGY

N.B.—In as much this Trīpraśnādhyāya has a good number of technical terms whose understanding is necessary to understand the Hindu methods of solving diurnal problems, we shall collect here all such technical terms under this heading for guidance.

Technical term	Meaning in modern terms	Symbol if any	Formula number	Occurs under verse	Hindu formula	Modern formula
Dṛk-jyā	Hindu sine of the Zenith-distance	H sin z	9	8	H sin z	R = 3438 R sin z
Dṛgamsacāpa	Zenith-distance	Z		,,		
Digjyā*	Hindu azimuth measured from the East point	H sin a		,,	H sin a	R sin a
Krantijyā	H sine of declination	H sin δ		,,	H sin δ	R sin δ
Akshajyā	H sine of latitude	H sin φ	15	,,	H cos φ	R cos φ
Lambajyā	H sine (colatitude)	H cos φ	16	,,	H cos φ	R cos φ
Chāyā	Gnomonic shadow	S	2	,,	$\frac{\text{K H sin z}}{\text{R}}$	K sin z
Chāyākarna	Hypot. of the △ one side being S	K		,,	$\sqrt{12^2 + S^2}$	

* Azimuth is called Digamsacāpa or Digamsam.

Technical term	Meaning in modern terms	Symbol if any	Formula number	Occurs under verse	Hindu formula	Modern formula
Chāyabhuja	Perpendicular from the extremity of the shadow on the East-west line.	b	3	8	$\frac{K\,H \sin z\,H \sin a}{R^2}$	$K \sin z \sin a$
Chayākoti	$\sqrt{s^2 - b^2}$		7	,,	$\frac{K\,H \sin x\,H \cos a}{R^2}$	$K \sin z \cos a$
Vishuvatchāya	Gnomonic shadow cast at mid-day on the equinoctial day at any place	s		,,	$\frac{12\,H \sin \varphi}{H \cos \varphi}$	$12 \tan \varphi$
Vishuvatkarṇa	Hypot. of the △, one side being s	k		,,	$\sqrt{s^2+12^2}$	$12 \sec \varphi$
Agrajyā	Hindu sine of rising azimuth	A	5	,,	$\frac{R\,H \sin \delta}{H \cos \varphi}$	$\frac{R \sin \delta}{\mathrm{Cos}\, \varphi}$
Karnāgra	Agrajyā reduced from a circle of radius R to one with radius K	a	6	,,	$\frac{K\,H \sin \delta}{H \cos \varphi}$	$\frac{K \sin \delta}{\mathrm{Cos}\, \varphi}$
Iṣta Śanku	The Hindu cosine of Z	I. S.	8	,,	$H \cos z$	$R \cos z$
Iṣta Hṛti	The hypot. of the △, one side being H cos z	I. H.	11	13–17	$\frac{R\,H \cos z}{H \cos \varphi}$	$\frac{R \cos z}{\mathrm{Cos}\, \varphi}$

Śankutala	The thrid side of the $\triangle$, above	S. T.	12	13–17	$\frac{H \cos z \, H \sin \delta}{H \cos \varphi}$	R cos z tan φ
Kujyā	The $\perp^{ar}$ distance bet. The lines drawn parallel to the east-west line through the rising point and the point of intersection of the diurnal circle and the great circle PEω		13/	,,	$\frac{H \sin \delta \, H \sin \varphi}{H \cos \varphi}$	R sin δ tan φ
Vishuvat-mandala	Celestial Equator					
Krānti Mandala	Ecliptic					
Kshitija	Horizon					
Unmandala	Great circle PEω or Equatorial horizon					
Yāmyottara-mandala	Meridian					
Samamandala	Prime-vertical					
Dṛk Mandala	Vertical					
Dhṛva Prota Vṛtta	Declination circle					

Technical term	Meaning in modern terms	Symbol if any	Formula number	Occurs under verse	Hindu formula	Modern formula
Kadamba Prota Vṛtta	Circle of celestial latitude					
Krānti	Hindu declination ie. arc of the declination circle intercepted bet. Ecliptic and Equator					
Spaṣta Krānti	Declination					
Dhṛvaka or Dhṛva	Celestial long measured from the Hindu zero point					
Sāyana Dhṛva	Celestial longitude					
Vikṣēpa	Polar latitude or the arc of the declination circle intercepted bet. the Ecliptic and the celestial body					
Sphuta Vikṣēpa	Celestial latitude					
Vishuvamsa Cāpa	Right ascension					

Antyā	The length of the $\perp^{ar}$ drawn from Q the culminating point of the celestial Equator on the line drawn parallel to the East-west line through the foot of the declination circle of the rising celestial body		14	13–17	$R + \frac{R\ H \sin\varphi\ H \sin\delta}{H \cos\varphi\ H \cos\delta}$	$R(1 + \tan\delta \tan\varphi)$
Sama-Śaṅku	H cos z when the body is on the prime-vertical	S. Ś.	17	20	$\frac{R\ H \sin\delta}{H \sin\varphi}$	$R \sin\delta / \sin\varphi$
Taddhṛti	Perpendicular distance between the lines drawn parallel to the East-west line through the rising point and the point where the diurnal circle cuts the prime-vertical		18	,,	$\frac{R^2\ H \sin\varphi}{H \cos\delta\ H \sin\varphi}$	$\frac{R \sin\delta}{\cos\varphi \sin\delta}$
Purvāparā	East-west line					
Udayāsta Sūtra	Line joining the rising and setting points					
Unmandala Śaṅku	H cos z when the body is on the Equatorial Horizon	U. S.	19	25	$\frac{H \sin\delta\ H \sin\varphi}{R}$	$R \sin\delta \sin\varphi$
Dinārdha Śaṅku	H cos z at the culminating point	D. S.	21	31–32	$H \cos(\varphi \underset{\sim}{+} \delta)$	$R \cos(\varphi \underset{\sim}{+} \delta)$

Technical term	Meaning in modern terms	Symbol if any	Formula number	Occurs under verse	Hindu formula	Modern formula
Yaṣti	The length of the ⊥ᵃʳ dropped from the culminating point on a plane parallel to the Horizon and passing thro' the point of intersection of the diurnal circle and Equatorial horizon	Y	21	33	$\frac{H\cos\varphi\ H\cos\delta}{R}$	$R\cos\varphi\cos\delta$
Hṛti	The line in the diurnal circle corresponding to Antiyā in the plane of the celestial Equator or the length of the ⊥ᵃʳ from the culminating point on Udayāstasūtra		22	3	$H\cos\delta+\frac{H\sin\delta\ H\sin\varphi}{H\cos\varphi}$	$R(\cos\delta+\sin\varphi\tan\varphi)$
Sūtra	OM (O=centre of the sphere M=foot of ⊥ᵃʳ on OQ from the foot of declination circle		26	53–54	$H\cos h$	$R\cos h$
Kalā	Corresponding line in diurnal circle		27	,,	$\frac{H\cos h\ H\cos\delta}{R}$	$R\cos h\cos\delta$
Iṣṭāntyā	Line corresponding to Iṣṭahṛti, on the Equatorial plane		28		$\frac{R^2\ H\cos z}{H\cos\varphi\ H\cos\delta}$	$\frac{R\cos z}{\cos\varphi\cos\delta}$

Cara Cāpa	Arc of the celestial Equator bet. the East point and foot of the declination circle					-
Carajyā	H sine of the above or the corresponding line of Kujya in the plane of the celestial Equator		13	13–17	$\frac{R \text{ H} \sin\varphi \text{ H} \sin\delta}{\text{H} \cos\varphi \text{ H} \cos\delta}$	R $\tan\varphi \tan\delta$
Natakāla	Hour angle	**h**				
Unnata	Time elapsed after rise					
Śara	Corresponding line of phala in the Equatorial plane		29	58	H vers (h)	R $(1-\cos h)$
Phala	⊥ar from the culminating point on a line through the celestial body parallel to Udayāstasūtra		30		$\frac{\text{H vers } h \text{ H} \cos\delta}{R}$	R $\cos\delta\,(1-\cos h)$
Ūrdhwa	Orthogonal projection of phala on the plane of the meridian		31	59	$\frac{\text{H vers } h \text{ H} \cos\varphi \text{ H} \cos\delta}{R^2}$	R$\cos\varphi\cos\delta(1-\cos h)$
Iṣṭayaṣṭi	The length of the ⊥ar dropped from the celestial body on a horizontaly plane passing through the top of U.S.	I. y	27/	55	$\frac{\text{H} \cos h \text{ H} \cos\delta \text{ H} \cos\varphi}{R^3}$	R $\cos\varphi \cos\delta \cos h$

Yet $\frac{R \sin \delta}{\sin \varphi}$ will have a value greater than R ie. even though the Sama-S'anku is never born so to say, it has a magnitude. 'तत्कथमिदं वन्ध्यासुतवत्' Bhāskara exclaims with respect to the magnitudes of Sama-S'anku and Taddhṛti as well, both of which are not there, yet, both of which have magnitudes greater than R. So he says "those magnitudes of the Sama-S'anku and Taddhṛti are like the sons of a barren lady". Then he says 'तद्पि प्रदृश्यते' ie. 'We shall show how they arise.' Here he uses his intuition of the principle of geometrical continuity. Even when the diurnal circle does not cut the prime-vertical, their planes intersect, outside the sphere and the perpendicular dropped from the point of intersection on the plane of the horizon is the Sama S'anku which has a magnitude greater than R. Similarly the Taddhṛti could be seen what it is now. These magnitudes can enter into a proportion like the I in verse 22, and do help us to get the other real magnitudes like the Unmandala S'anku etc.

Verses 66, 67 *and first half of* 68. To obtain the time from the shadow.

$$\text{Iṣṭāntyakā} = \frac{\text{U.K.} \times \text{Carajyā}}{\text{I.K.}} = \frac{\text{D.K.} \times \text{Antyā}}{\text{I.K.}}$$

$$= \frac{R^2}{H \cos \delta \times \text{I.K.}}$$

Dinārdha $- h =$ Unnatakāla

where K $=$ Karṇa, $k =$ Vishuvat Karṇa, I.A. $=$ Iṣṭāntya.

Comm. We had under verse 62.

Iṣta-Karṇa $\times$ Iṣta-Hṛti $=$ D. K. $\times$ Hṛti $=$ S. K. $\times$ Taddhṛti $=$ U. K. $\times$ Kujyā multiplying throughout by $\frac{R}{H \cos \delta}$

Iṣṭa-Karṇa × I. A. = D. K. × Antya = U. K. × Carajyā

so that $\text{I.A.} = \dfrac{\text{D.K.} \times \text{Antyā}}{\text{I.K.}} = \dfrac{\text{U.K.} \times \text{Carajyā}}{\text{I.K.}}$ I

But $\text{U.K.} = \dfrac{12\text{ R}}{\text{U.S.}}$ (verse 40)

$$\therefore \text{ U.K.} \times \text{Carajyā} = \frac{12\text{R} \times \text{Carajyā}}{\text{U.S.}}$$

$$= \frac{12\text{R} \times \text{Kujyā} \times \text{R}}{\text{U.S.} \times \text{H} \cos \delta} \text{ since Carajyā} = \frac{\text{Kujyā} \times \text{R}}{\text{H} \cos \delta}$$

But Kujyā and U.S. are the Karṇa and Koṭi of the seventh latitudinal triangle so that $\dfrac{\text{Kujyā}}{\text{U.S.}} = \dfrac{k}{12}$

Comparing with the first fundamental latitudinal triangle.

$$\therefore \text{ U.K.} \times \text{Carajyā} = \frac{12\text{R}^2 \times k}{\text{H} \cos \delta \; 12} = \frac{k\text{R}^2}{\text{H} \cos \delta}$$

Hence substituting in I

$$\text{I.A.} = \frac{\text{U.K.} \times \text{Carajyā}}{\text{I.K.}} = \frac{k\text{R}^2}{\text{H} \cos \delta \times \text{I.K.}}$$

Thus we have proved the first part of the statement. Having obtained I.A., from fig. 52 we have Antyā — I.A. =(FQ−AN)=CQ. The Utkrama Cāpa of CQ = NQ = h and Dinārdha — h = Unnatakāla. Let us see what this procedure means in practice. Since on any day at any place, φ and the declination of the Sun are known we can compute all the magnitudes given in the verse or more easily H cos δ so that from the formula Iṣṭāntyā = $\dfrac{12\text{R}^2}{\text{H} \cos \delta \times \text{I.K.}}$ where $\text{K} = \sqrt{\text{S}^2 + 12}$, the shadow being observed Iṣṭāntya could be computed in no time. Also the Antyā of the day R+(H tan φ tan δ) can be computed so that the segment CQ can be got. The inverse Hversine of this is h. The arc CQ above was symbolized as S'ara.

Bhāskara's proof of I.A. $= \frac{kR^2}{H\cos\delta \times I.K.}$ proceeds from first principles as follows:—(i) If by k we have 12 as the Koṭi what have we for R? The result is H cos z, Mahās'anku. ∴ Mahā S'anku $= \frac{12R}{I.K.}$. From Maha S'anku we pass on to Iṣṭa-Hṛti with which it forms a latitudinal triangle. If by the gnomon of 12 units we have k the Vishuvat Karṇa, what have we by Mahā S'anku? The result is $\frac{12R}{I.K.} \times \frac{k}{12} = \frac{kR}{I.K.}$. Again from the Iṣṭa-Hṛti we pass on to I.A. by multiplying by $\frac{R}{H\cos\delta}$ so that I.A. $= \frac{kR}{H\cos\delta \times I.K.}$ as given.

Second half of verse 68. The inverse Hversine if a quantity x greater than R, is $5400 + H\sin^{-1}(\theta)$ when $x - R = \theta$.

Comm. Since Hvers $(90+\theta) = R + H\sin\theta = x$ (say)
$90 + \theta = \text{Hvers}^{-1}(R + H\sin\theta) =$

But 90° are equal to 5400 asus and $\theta = H\sin^{-1}(H\sin\theta)$ $= H\sin^{-1}(x - R) =$ त्रिस्याधि कभागस्यक्रमचापम्

∴ 5400 + त्रिज्याधिकभागक्रमचापम् = Utkrama Cāpa of a quantity.

Verse 69. Alternate method of obtaining the time that has elapsed after Sunrise noting the shadow S.

Subtract Charajyā from or add it to Iṣṭāntyā according as δ is north or south. Obtain inverse H sine of the remainder and add the Caracāpa to this inverse H sine. Then we have the unnatakāla by converting the result into time.

Comm: Ref. fig. 52. I.A. = AN. I.A. – Carajyā = B.N. Hvers^{-1} (BN) = arc EN. Arc EN + Caracāpa = EN + EM = MN. This converted into time is evidently the Unnatakāla because the arc MN of the equator is the arc intercepted between the feet of the declination circles at rising and at the time concerned M being the foot of the rising declination circle and N the foot of that at the time in question. The convention of signs is clear.

Verse 70 *and first half of* 71. To obtain the Sun's longitude from the shadow S.

The gnomonic shadow at noon, being multiplied by R and divided by K, the inverse H sine of the result gives the meridian zenith-distance. This being decreased or increased by the latitude gives the Sun's declination according as the extremity of the shadow is north or From the declination, we have the Sun's longitude by the formula $H \sin \delta = \frac{H \sin \lambda \; H \sin \omega}{R}$

Comm. We have from the triangle formed by the gnomon and the shadow S, $\frac{S}{K} = \sin z$ or $\frac{SR}{K} = H \sin z$

$\therefore H \sin^{-1}\left(\frac{SR}{K}\right) = z$. Since we are directed to take the mid-day shadow, we have the meridian zenith-distance and from the formula $z + \delta = \phi$ we have δ. If the extremity of the shadow be north, the Sun is south of the zenith, and then $\phi \sim z = \delta$. The word वियुक्ता: is used to signify difference which is positive. If $z > \phi$ then the declination is south and vice-versa. If the extremity of the shadow is on the south, the Sun is on the north of the zenith. in which case $\phi + z = \delta$.

Second half of verse 71. To obtain ϕ from δ.

If the zenith distance and the declination are of the same direction, their difference, otherwise their sum will be the latitude.

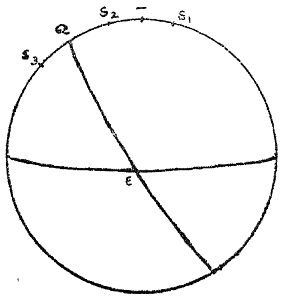


Fig. 58

Comm. Suppose the zenith-distance is north and declination also north, then clearly from fig. 58, in this position S_1 of the Sun $QS_1 - ZS_1 = \phi = \delta - Z$ (1) Again in the position S_3 of the Sun, zenith-distance is south and declination is also south; so, here also difference gives ϕ ie. $ZS_3 - QS_3 = Z - \delta = \phi$. (3) In the position S_2, however, Z is south and δ is north, so that their sum is equal to ϕ. It will be noted that the Hindu convention of signs does not contemplate negative declination and also it uses the word 'difference' to signify the positive difference, as for example, in the first two cases $\delta \sim Z$ is taken as ϕ. The modern formula $Z + \delta = \phi$ applies universally with the convention that δ is +ve if north, ϕ is +ve if south.

Verses 72, 73. To obtain the Bhuja from the shadow. Karṇa Vrittāgrā $= \dfrac{A \times K}{R}$ where A = Agrā and K the Chayākarṇa. This Karnāgrā is to be taken as belonging to the opposite hemisphere to the Sun. Calling this Karnāgra as a and the equinoctial shadow as s, $a \pm s$ according as δ is south or north gives the Chāyābhuja b. Thus $a \pm s = b$. If the extremity of the shadow be on the north and δ be north, then $b + a = s$; if δ be south $b \sim a = s$. If b be north, $b \sim s =$ A, otherwise ie. if south $b + s = a$. $\dfrac{R \times a}{K} =$ Agrā and

$$\frac{\text{Agrā} \times \text{Koṭi of latitudinal triangle}}{\text{of the latitudinal triangle}} = \quad \text{sin}$$

Comm. These verses are very important and the contents have been already elucidated under verses 13–17. We shall elucidate the convention of signs in more detail both from the modern point as well as from the Hindu traditional point. First we shall discuss the modern. We have the formula A = S + B where A = Agrā, S = S'ankutala and B, S'anku-bhuja. Let us confine ourselves to north latitudes alone, for, south latitudes did not concern the Hindu astronomers. Then treating north declination as positive and also north Hindu azimuth as positive the above formula holds universally. (Ref. fig. 59) Let the figure represent the meridian plane. Let S_1, S_2, S_3 be the projections of the Sun's positions in their diurnal circles on to the meridian plane. Let A, B. be the

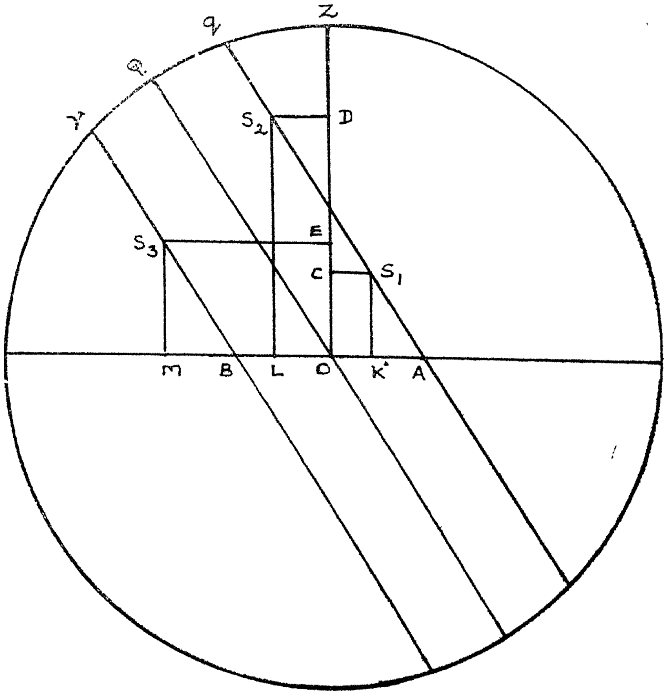


Fig. 59

projections of the rising points of the Sun on the same plane. Let perpendiculars be dropped from S_1, S_2, S_3 on the plane of the horizon. Let *o* be the centre of the sphere. Let *ns* be the north-south line. In position S_1, S_1C = S'anku-Bhuja, KA = S'ankutala, *o*A = Agrā, so that A=S+B (1) In the position S_2. S_2D = S'anku-Bhuja, LA = S'anku-tala, *o*A = Agrā, so that B + A = S ; but here B is negative, a the azimuth being south so that writing − B for B, − B + A = S ∴ A = S + B again. In position S_3, OB=A, MB=S, OM=B so that A+S=B ; but here A is +ve, δ being south and B is negative a being south ; hence writing − A and − B for A and B A + S = − B or A = S + B again. This shows that with the convention cited above A = S + B holds good universally.

Now let us consider the situation with respect to the Karnāgrā. Each of the three quantities *a*, *b*, *s* have now opposite directions. If δ be north, the Sun will be on the north of Equator, but the extremity of the shadow will now be on the south of the Equinoctial shadow line (E.S.L.) ie. the line which is parallel to the East-west line at a distance of the equinoctial shadow *s*, and which is the locus of the extremity of the shadow on the equinoctial day ; thus when the Agrā is considered to be positive being on the north of the East point, the Karnāgrā, though it is on the south will have to be considered positive. Similarly when the azimuth of the Sun *a* is on the north of the prime-vertical and is so considered to be positive, the extremity of the shadow being on the south of the East-west line and has a negative azimuth, the bhuja is still to be considered positive. Again the S'ankutala being always south of the Udayāsta Sūtra being considered south and positive, the corresponding quantity into which it gets converted on the horizontal dial namely the equinoctial shadow *s* will be north of the East-west line and will be considered

positive. In other words in the equation $a = b + s$, a is positive when δ is north, b is +ve when the azimuth of the Sun is north of the East point, and s is always positive. Since in north latitudes, S'ankutala will be always south of the Udayāstasūtra and considered positive, the E.S.L. will be on the north of the East-west line so that s is considered positive. We should have had to consider s negative in southern latitudes, as per the above convention but as the Hindu astronomers did not have to concern themselves with south latitudes, the question of sign for s did not arise except taking it as always positive. Hence the equation $a = b + s$ will hold universally with the same conventions of sign which we stipulated with respect to the equation A = B + S. The foregoing analysis is on the modern lines.

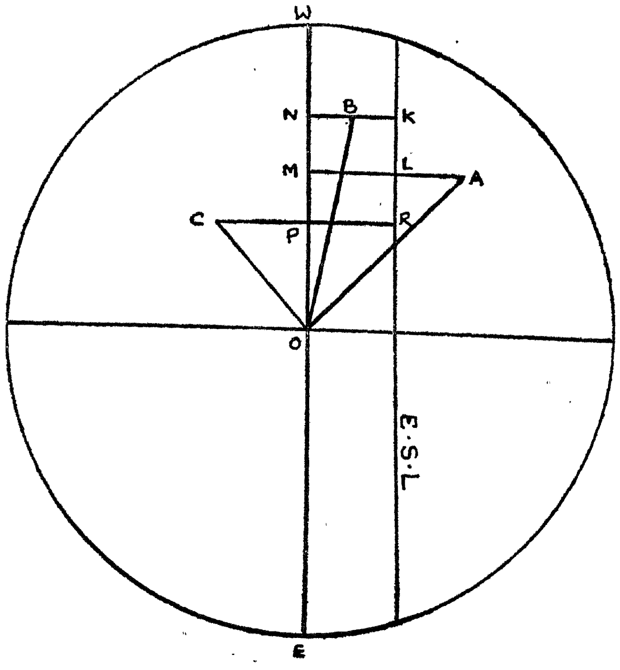


Fig. 60

Now let us see how the convention of signs is stipulated in Hindu astronomy with respect to the equation $a = b + s$. We have said that 's' is always north of the East-west line and considered positive. Regarding 'a', it is said by Bhāskara व्यस्तगोला which means that when δ is north and the Sun is said to be in northern hemisphere, $\underline{a}$ is said to belong to the southern hemisphere. Also when the Sun is on the north of the prime-vertical and S'ankubhuja is considered north, the Chāyābhuja being south of the East-west line is considered south. With these conventions of directions (we say of directions, and not signs because the Hindu astronomers do not speak of signs but only of directions) it is stipulated in Hindu astronomy that quantities of like directions are to be added, otherwise their difference is to be taken ोगः, भिन्नदिशोः अन्तरम्'. This convention stipulating addition or difference is technically called ' Samskāra '. That is why it is said simply ie. 'Samskāra (on the aforesaid lines) is to be effected between a and s to get bhuja b'. Here it may be reiterated that the word 'अन्तरम्' ie. 'difference' is used in its restricted sense namely that the positive difference alone is to be taken. Thus the 'antaram' of 8 and 5 is 3 as well as that of 5 and 8 is also 3. Then it might be asked how to decide the direction of the bhuja, if we were to take s as equal to $a \sim b$ and not $a-b$ or $b-a$. The answer is that between $\underline{a}$ and $\underline{s}$ whose directions are known as per the aforesaid convention, equating their difference ie. $a \sim s$ to b, we have to take b as having that direction which is indicated by that quantity either a or s which has a larger numerical value.

Thus while on modern lines we take $a = b + s$ to hold universally with the convention of signs which we have agreed to on modern lines namely that a is +ve if δ is north and b is positive if the Hindu azimuth is north of the East point and s is always +ve, we take on the Hindu lines $a \underset{\sim}{+} s = b$ with the conventions stipulated with respect

to directions (not of signs) namely that a is south if δ is north, s is always north and b is to be taken to belong to that direction to which the numerically bigger quantity of a and s belongs in the case of difference. Also it is to be taken to belong to that direction of a and s when both of them have the same direction.

With this convention in mind, Bhāskara clarifies the convention by citing examples.

(1) S=5, δ is north, Agrā=916′-48″ K=30 so that $a = \frac{KA}{R} = \frac{916\frac{4}{5} \times 30}{3438} = 8$ units (south, because it should be taken to be व्यस्तगोला ie. belonging to the direction opposite to that of δ).

Question. "What is b and of what direction?"

Answer. $b = a \underset{\sim}{+} s = 8$ (south) $\sim$ 5 (north). (We are taking the difference because Samskāra is to be construed as addition of quantities having the same direction and difference of quantities of opposite direction $\therefore$ $b = 3$ and *is on the south* because the numerically bigger quantity of 8 and 5 belongs to south.

Q. 2. δ is north S=5, A=916′-48″; K=15, 'what is b and in what direction?'

Answer. $a = \frac{KA}{R} = 4$; $b = a \underset{\sim}{+} s = 4 \underset{\sim}{+} 5 = 4 \sim 5$ (here a is south δ being north and s is north so that difference is stipulated as above) $= 1$ (north because 5 belongs to north.

We add here two more examples to illustrate addition by saying that δ is south in the above examples so that a is north in both the examples. Hence in (1) $b=8+5=13$ (north) and in (2) $b=4+5=9$ (north). Refering to Fig. 60, we see there three cases depicted namely the extremi-

ties of the shadows being A, B and C. In the first case AL = Karnāgra=a (Karnāgra is the distance of the extremity of the shadow from the E.S.L. namely K.L.R. whereas bhuja is the distance of the same from the East-west line namely PMN) AM = bhuja and ML = s so that $b=a+s$ addition being justified since both a and s are of the same direction namely north. In the second case BK = Karnāgra, BN = bhuja and NK = s so that $b=s-a$, the difference being justified because a is south and s north. Here we have taken the difference as $s-a$ and not $a-s$ because s is numerically greater and being oriented north, the bhuja is north. In the third case, PR=s, CP=b and CR=a so that $b=a-s$, the difference being justified because s is north and a south. Also we have taken the difference as $a-s$ and not $s-a$ because a is numerically greater and as such lends its direction namely 'south' to the bhuja. Thus in the three examples cited, addition is prescribed between a and s only when the extremity of the shadow is to the north of E.S.L. In the case of A=S+B or B=A−S also, addition is prescribed only when δ is south, which means that the corresponding a ie. AL is north. In fact the prescription of addition or difference accord in the cases of both the equations either

Verses 74 *and* 75. Hereafter questions are being set and answered on diurnal problems.

Seeing the shadow of the gnomon, the azimuth and longitude of the Sun or seeing two shadows with their respective directions, whoever knows the equinoctial shadow of the place, I consider him as the Garuda or Eagle who could overcome the false pride of puffed up snakes of astronomers.

Given that when K=30 units, the bhuja is 3 units south, and when K=15, the bhuja is 1 unit north, compute the latitude, or again given H sin δ = 846 and given K and b of a shadow, compute the equinoctial shadows.

Comm. The questions are clear the second verse illustrating the first.

Verse 76. Answer of the first question.

$\frac{b_1 K_2 \overset{\sim}{+} b_2 K_1}{K_2 \sim K_1} = s$ according as the bhujas are of the same or opposite directions.

Comm. Suppose b_1 and b_2 are of the same direction so that $b_1 = a_1 - s$ and $b_2 = a_2 - s$, taking the modern convention of signs. But $a_1 = \frac{K_1 A}{R}$ and $a_2 = \frac{K_2 A}{R}$

$$\therefore \quad b_1 = \frac{K_1 A}{R} - s \text{ and } b_2 = \frac{K_2 A}{R} - s$$

$$\therefore \quad \frac{b_1 + s}{K_1} = \frac{A}{R} = \frac{b_2 + s}{K_2}$$

$$\therefore \quad K_2 b_1 - K_1 b_2 = s (K_1 - K_2)$$

$$\therefore \quad s = \frac{K_2 b_1 - K_2 b_2}{K_1 - K_2}$$

If, however b_1 and b_2 are of opposite directions ie. of opposite signs, writing $-b_2$ for b_2, we have

$s = \frac{K_2 b_1 + K_1 b_2}{K_1 - K_2}$. Here we have chosen to follow the modern convention of signs; otherwise we have to consider four alternatives, for, bhujas of the same direction might mean both of the type OC (fig. 60) or both of the type of OB or one of the tyye OB and one of the type of OA or both of the type OA.

Verses 77 *and* 78. Answer to the second question.

Let Laghu $\equiv L = \left(\frac{KH \sin \delta}{R}\right)^2$; $12^2 (L - b^2) \equiv$ Ādya; Para $= 12^2 b$. Let Ādya and Para be divided by $L \sim 12^2$; call them still Ādya and Para; then $\sqrt{Para^2 + A} \pm Para$ $= s$ according as b is north or south.

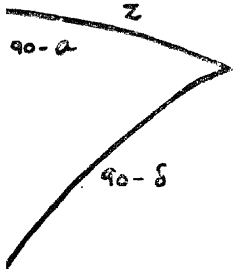


Fig. 61

Comm. The data are H sin δ, K and b; since K and b are given a is known. Thus from the triangle PZS (fig. 61) we have sin δ = sinϕ cos z+cosϕ sin z sin a, all quantities except ϕ are known. Solving this trigonometrical equation which is of the form $a \cos \phi + b \sin \phi = c$, we can have ϕ.

We shall now see how it is solved by Bhāskara. Let s be the equinoctial shadow. Then

$$a = b + s = \frac{\text{KA}}{\text{R}} = \frac{\text{KH} \sin \delta}{\text{H} \cos \phi} \qquad \text{I}$$

But $\frac{12}{\text{K}} = \frac{12}{\sqrt{s^2 + 12^2}} = \frac{\text{H} \cos \phi}{\text{R}}$ so that

$\text{H} \cos \phi = \frac{12\text{ R}}{\sqrt{s^2 + 12^2}}$. Substituting in I

$$b + s = \frac{\text{KH} \sin \delta \sqrt{12^2 + s^2}}{12 \text{ R}}$$

which reduces to $12^2 \text{R}^2 (b + s)^2 = \text{K}^2 \text{ H} \sin^2 \delta \, (12^2 + s^2)$

ie. $s^2 (12^2 \text{R}^2 - \text{K}^2 \text{ H} \sin^2 \delta) + 2b\, 12^2 \text{R}.^2 s =$
$12^2 \{(\text{H} \sin^2 \delta) \text{K}^2 - b^2 \text{R}^2\}$

ie. $s^2 \left(12^2 - \frac{\text{K}^2 \text{ H} \sin^2 \delta}{\text{R}^2}\right) + 2.\, 12^2 b.\, s) = 12^2$
$\left(\frac{\text{K}^2 \text{ H} \sin^2 \delta}{\text{R}^2} - b^2\right)$

Here $\frac{\text{K}^2 \text{ H} \sin^2 \delta}{\text{R}^2}$ is symbolized as L

$12^2 \left(\frac{\text{K}^2 \text{ H} \sin^2 \delta}{\text{R}^2} - b^2\right)$ put as Ādya and $12^2\, b$ is put as para.

So the equation reduces to
$s^2(12^2 - L) + 2.$ Para $s = \bar{A}$dya ; Divide throughout by $12^2 - L$ and put again $\frac{\text{Para}}{12^2 - L}$ as Para and $\frac{\bar{A}\text{dya}}{12^2 - L}$ $\bar{A}$dya Then the equation reduces to $s^2 + 2$ Para $s = \bar{A}$dya ; completing the square $(s + \text{Para})^2 = \text{Para}^2 + \bar{A}\text{dya}$

$$s + \text{Para} = \sqrt{\text{Para}^2 + \bar{A}\text{dya}}$$

$$s = \sqrt{\text{Para}^2 + \bar{A}\text{dya}} - \text{Para}$$ as one solution.

We have taken to start with $a = b + s$ which holds good according to the Hindu convention when b is south ; if, however b is north $a = b \sim s$ so that $(b \sim s)^2 = s^2 + b^2 - 2bs.$ So in the equation we have to write $-s$ for s, so that we have now $s^2 - 2$ Para $s = \bar{A}$dya ie. $(s - \text{Para})^2 = \text{Para}^2 +$

$\therefore \quad s = \sqrt{\text{Para}^2 + \bar{A}\text{dya}} + \text{Para}$ as the second solution.

Verse 79. When the Sun's longitude is 135°, the shadow of the gnomon is 12 units and west. What is the latitude?

Comm. Here is a method of obtaining the latitude of the place by observing the gnomon's shadow when the Sun is on the prime-vertical.

Verse 80. Answer to the question above.

$$\frac{12\text{ R}}{\text{K}} = \text{H} \cos z\,;$$

$$s = \frac{12\text{ H} \sin \delta}{\sqrt{\text{Sama-S'anku}^2 - \text{H} \sin^2 \delta}}$$

Comm. Solution in modern terms.

$S = 12 \quad \therefore \tan z = 1 \quad \therefore z = 45$; but when the Sun is on the prime-vertical, we have by Napier's rule

$\text{Sin}\,\delta = \sin \phi \cos z = \sin \phi\, \sqrt{2}$. But since $\lambda = 135°$

$\text{Sin}\ \delta = \sin \lambda \sin \omega = \sin 135 \sin \omega = \sqrt{\bar{2}} \sin \omega$

$\therefore \quad \sin \phi \sqrt{\bar{2}} = \sqrt{2} \sin \omega \qquad \therefore \quad \omega = \phi$

$\therefore \ \tan \phi = \tan 24°$

$= .4452 \quad \therefore \quad s = 12 \times .4452 = 5.3424 = 5'' - 20'''$

Bhāskara's solution. Taking the fifth latitudinal triangle $\frac{\text{H} \sin \delta}{x} = \frac{s}{12}$ where x is the Koṭi

$\therefore \quad s = \frac{12 \text{ H} \sin \delta}{x}$. But we are given that the Sama-S'anku ie.

$\text{H} \cos z = \text{R} \cos z = \frac{\text{R}}{\sqrt{2}}$ because $z = 45°$ when the shadow equals the length of the gnomon.

and $\quad \text{H} \sin \delta = \frac{\text{H} \sin \lambda \text{ H} \sin \omega}{\text{R}} = \frac{\text{H} \sin 135 \text{ H} \sin \omega}{\text{R}}$

$= \frac{\text{H} \sin 45 \text{ H} \sin \omega}{\text{R}} = \text{R}/\sqrt{2} \times \frac{\text{H} \sin \omega}{\text{R}} = \frac{\text{H} \sin \omega}{\sqrt{2}}$

$\therefore x = \sqrt{\text{Sama-S'anku}^2 - \text{H} \sin^2 \delta} = \sqrt{\frac{\text{R}^2}{2} - \frac{\text{H} \sin^2 \omega}{2}}$

$= \frac{\text{H} \cos \omega}{\sqrt{2}} \quad \therefore \quad s = \frac{12 \text{ H} \sin \delta \times \sqrt{2}}{\text{H} \cos \omega} =$

$\frac{12 \text{ H} \sin \omega}{\sqrt{2}} \times \frac{\sqrt{2}}{\text{H} \cos \omega}$

$= \frac{12 \text{ H} \sin \omega}{\text{H} \cos \omega} = \frac{12 \times 1397}{\sqrt{3438^2 - 1397^2}}$

since $\text{H} \sin \omega = 1397$ (verse 12 Spaṣṭādhikāra)

$= 5'' - 20'''$

Verse 81. Two more questions.

An observer at Ujjain observed that the Sun was on the prime-vertical 5 nādis after Sun-rise or 5 nādis afternoon. If you could give me the declination of the Sun at

that moments I would reckon you as one who could be well compared with the goad that could be applied to the head of the wild elephants of puffed up astronomers.

Verses 82, 83. Answer to the first question.

Assume the H sine of the Unnatakāla to be Iṣṭa Hṛti in the first place. Multiply it by 12 *s* and divide by k^2 the square of the Viṣuvatkarṇa. Then you get an approximate value of H sin δ. Then compute with this, H sin δ, Charajyā etc. and thereby obtain a more correct value of the Iṣṭa Hṛti. Multiply this by the H sin δ got before and divide by the first Hṛti assumed. Then we have a nearer approximation of H sin δ. Repeat the process till a stationary value has been reached. That will be the correct H sin δ.

Comm. We know that H sine of the Unnatakāla is nearly the Iṣṭāntyakā. It will be noted that Iṣṭāntyakā is the sum of two H sines namely (1) Carajyā (2) H sine of Unnatakāla minus Chara. The second H sine is called Sūtra or H cos *h*. (Vide page 278).

Thus *Sūtra* + *Carajyā* = *Iṣṭāntyakā* whereas H sine (Unnatakāla) = H sine of the Cāpas of Carajyā and Sūtra. In other words H sin (Unnatakāla) = H sin (H $\sin^{-1}$ Carajyā + H $\sin^{-1}$ (Sūtra)). Iṣṭa Hṛti is Iṣṭāntyaka $\times \dfrac{\text{H} \cos \delta}{\text{R}}$. But we do not know H sin δ so that Iṣṭa Hṛti could not be got. So we will not be far from truth in assuming the given Unnatakāla to be Iṣṭa Hṛti itself ie. Taddhṛti here, as the Sun is on the prime-vertical. The formula for Taddhṛti is

$$\frac{\quad}{\text{H} \cos \phi \text{ H} \sin \phi} = \frac{\text{R}^2 . \text{ H} \sin \delta}{\frac{\text{R. } 12}{k} \times \text{R} \times \frac{s}{k}} = \frac{\text{K}^2 \text{ H} \sin \delta}{12\, s}$$

$\therefore \quad \frac{\text{Taddhṛti} \times 12\,s}{k^2} = \sin \delta$. Thus assuming H sine of the given Unnatakāla to be Taddhṛti and multiplying it by 12 s and dividing by k^2 we have the value of H sin δ. But this is approximate because the given Unnatakāla is not exactly Taddhṛti but only an approximate value. From this H sin δ, compute H cos δ, Charajyā, Kujyā and through the process indicated in verse 54 namely "Subtract the Characāpa from the Unnatakāla. The H sine of the result is called Sūtra. Multiply the Sūtra by H cos δ and divide by R; then we have Kalā. Add Kujyā to Kalā; we get Iṣṭa-Hṛti", we obtain a more correct value of Taddhṛti. Then here we may cut short the process as follows namely 'If by the assumed Taddhṛti we had the previous H sin δ, what shall we have for this more approximate Taddhṛti'? The result will be a more approximate value of H sin δ. Again form the Taddhṛti with this H sin δ and so repeating the process till we have a stationary value, we have the correct value of H sin δ.

Note. This is a beautiful example of the method of successive approximations which is a modern technique but which was so much in vogue and favourite with the Hindu astronomers. (It will be noted how to cut short the method).

Verses 84, 85. Answer to the second question.

Obtain $12^2\, R^2/(R^2 - H \sin^2 h)\, s^2 + 1$ and divide R^2 by this and take the square root which gives H sin δ. Then $\frac{R \times H \sin \delta}{H \sin \omega}$ gives H sin λ whose Cāpa gives the longitude of the Sun.

Comm. The H sine of the given Natakāla is H sin h and $R^2 - H \sin^2 h = H \cos^2 h$. Let H sin δ be x, which is required to be found. Then $R^2 - x^2 = H \cos^2 \delta$; H cos h = Sūtra and

$$\frac{\text{Sūtra} \times \text{H} \cos \delta}{\text{R}} = \text{Kalā} = \frac{\text{H} \cos h \, \text{H} \cos \delta}{\text{R}} = \frac{\text{H} \cos h \sqrt{\text{R}^2 - x^2}}{\text{R}}$$

But Kalā is the Koti of the fifth latitudinal triangle of which H sin δ is Bhuja. Hence $\dfrac{\text{Kalā} \times}{\text{H} \cos} = \text{H} \sin \delta = x$

ie. $\dfrac{\text{H} \cos h \sqrt{\text{R}^2 - x^2}}{\text{R}} \times \dfrac{\text{H} \sin \phi}{\text{H} \cos \phi} = x$; but $\dfrac{\text{H} \sin}{\text{H} \cos \phi}$

$\therefore$ Squaring both sides $\dfrac{(\text{R}^2 - \text{H} \sin^2 h)(\text{R}^2 - x^2)}{\text{R}^2} \times \dfrac{s^2}{12^2} = x$

$\therefore \quad 12^2 \text{R}^2 x^2 = s^2 \text{R}^2 (\text{R}^2 - \text{H} \sin^2 h) - s^2 x^2 (\text{R}^2 - \text{H} \sin^2 h)$

ie. $x^2 \{(12^2 \text{R}^2 + s^2 (\text{R}^2 - \text{H} \sin^2 h)\} = s^2 \text{R}^2 (\text{R}^2 - \text{H} \sin^2 h)$

$$x^2 = \frac{s^2 \text{R}^2 (\text{R}^2 - \text{H} \sin^2 h)}{12^2 \text{R}^2 + s^2 (\text{R}^2 - \text{H} \sin^2 h)}$$

$$\frac{\text{R}^2}{\dfrac{12^2 \text{R}^2}{s^2 (\text{R}^2 - \text{H} \sin^2 h)} + 1} \quad \therefore x = \sqrt{\frac{\text{R}}{\dfrac{12^2 \text{R}^2}{s^2 (\text{R}^2 - \text{H} \sin^2 h)}}}$$

H sin δ as given. From H sin δ, the method of obtaining λ is clear from the formula $\dfrac{\text{H} \sin \lambda \, \text{H} \sin \omega}{\text{R}} = \text{H} \sin \delta$. In the given numerical example $h = 5$ nādis $= \dfrac{360^\circ}{12} = 30^\circ$ since 60 nadis of time correspond to 360°.

Thus H sin $h = \dfrac{\text{R}}{2}$; the remaining work follows.

Verse 86. Another question.

When the Sun is on the prime-vertical the gnomonic shadow is noted to be 16 inches. The Unnatakāla is 8 nādis. If you could give the H sin δ and s, I shall consider you nothing short of one who is an adept in solving the totality of the diurnal problems.

Verses 87 and 88. Answer to the question.

Here also assume H sin (Unnata) to be the Taddhṛti as formerly done. Then as the shadow is 16″,

$K = \sqrt{16^2 + 12^2} = 20''$. Then $H \cos z = \frac{12\ R}{K} = \frac{12}{20} \times 3438$

Unnatakāla = 8 nādis = 48°.

$\therefore$ H sin (48°) = assumed Taddhṛti. Then from the fourth latitudinal triangle, $\frac{H \sin 48}{H \cos z} = \frac{k}{12}$

$\therefore$ Approximate value of k is $\frac{12\ H \sin 48}{12/20 \times 3438} = \frac{20\ H \sin 48}{3438}$.

This is a known quantity from which s could be computed since $12^2 + s^2 = k^2$. Again from the fifth latitudinal triangle $\frac{s}{k} = \frac{H \sin \delta}{S.\ S.}$.

Here s, k and S. S. (Samamandala-S'aṅku) are known $\therefore$ H sin δ could be got approximately. Thus we have found approximately the required quantities s and H sin δ. From this H sin δ and s we have to compute again H cos δ, Carajyā, Kujyā, etc. and applying the procedure of verse 54

$H \sin \frac{(\text{Unnata-cāra}) \times H \cos \delta}{R}$ + Kujyā = Iṣṭa Hṛti; this is nearer value of Iṣṭa Hṛti than the assumed Taddhṛti. From this again obtain as before s and H sin δ; we could not apply the proportion " If by the assumed Taddhṛti we have the previous H sin δ, what shall we have for this computed Taddhṛti (Iṣṭa Hṛti) " for the reason given below. So Repeat the entire process till an invariable quantity is got which will be the correct value of H sin δ.

Here repeating the entire calculation is correct and not taking the proportion because Taddhṛti $= \frac{R \sin \delta}{\sin \phi \cos \phi}$ where $\sin \delta$ and $\sin \varphi$ are both to be computed.

Formerly we could take the proportion in verse 81 because the latitude of Ujjain being known, in the magni-

tude of Taddhṛti namely $\frac{R\ H \sin \delta}{\sin \phi \cos \phi}$ only H sin δ is variable and Taddhṛti is directly proportional to H sin δ. But in the pre ent example both H sin δ and H sin ϕ are both variables so that, that kind of rule of three does not work.

Verse 89. Oh! Mathematician! At a place where $s = 5''$, there 10 nādikas after Sun-rise the shadow S is observed to be 9″. Tell me what the longitude of the Sun would be, if you are an adept in computing as well as understanding the geometry of the sphere.

Verses 90, 91. Answer to the question posed.

Assume H sine (Unnatakāla) to be Iṣṭāntyakā. Then $\frac{K \times H \cos z \times R}{12 \times I.\ A.}$ H cos δ where I. A. = Iṣṭāntyakā. $R^2 - H\cos^2\delta = H\sin^2\delta$; from this approximate H sin δ and the given s compute a more approximate I. A. Repeat the process till an invariable quantity is obtained for H sin δ, which will be its correct value. From this, using the formula H sin $\delta = \frac{H \sin \lambda\ H\ si}{R}$

Comm. We know the formula for I. A. as

$\frac{R^2\ H \cos z}{H \cos \varphi\ H \cos \delta}$ Assuming Unnatakālajyā as I. A.

$$H \sin (\text{Unnatakāla}) = \frac{R^2\ H\ \text{co}}{H \cos \varphi\ H \cos \delta}$$

$$\frac{R^2\ H \cos z}{\frac{12\ R.\ I.A.}{k}} = \frac{k \times R \times H \cos z}{12 \times I.\ A.};\ \text{from}$$

which obtaining H sin δ and proceeding as indicated we have λ. In the above proof we have used our formula. But Hindu Astronomers proceed from first principles. Let us hear Bhāskara. Since $S = 9''$; $K = \sqrt{9^2 + 12^2} = 15''$

$\therefore$ Mahā-S'anku $=$ H cos z $= \frac{R \times 12}{K} = \frac{3438 \times 12}{15} = 2750-24.$

We know that Mahā-S'anku forms a latitudinal triangle with Iṣta Hṛti.

$$\text{So } \frac{\text{H cos z}}{\text{Iṣta Hṛti}} = \frac{12}{k} \quad \therefore \text{ Iṣta Hṛti} = \qquad 12$$

$$\text{Iṣtāntyā} = \frac{\text{Iṣta Hṛti} \times R}{\text{H cos } \delta} \text{ ie. H cos } \delta = \frac{\qquad}{\text{Iṣtāntyā}}$$

$= \frac{\text{H cos z} \times k \times R}{12 \times \text{I. A.}}$ substituting the above value of Iṣta Hṛti. Here H cos z is got above and I. A. has been assumed above as H sin (Unnatakāla).

Note. (1) Computing

$$= \frac{12 R}{15} \times \frac{\sqrt{s^2 + 12^2} \times}{12 \times \text{H sin (60)}}$$

$\frac{R^2 \times 13}{15 \text{ H sin } 60} = \frac{13 R \times 2}{15 \times \sqrt{3}}$. Here H cos $\delta > R$ which is invalid.

(2) This is the only place where Bhāskara gave a numerical example with a slight flaw. In other words, under the given circumstances the shadow must be greater than what is given. However, the procedure indicated is mathematically correct.

(3) It is interesting to note that the flaw was noted by a commentator named Lakṣmidāsa as reported by Munīs'wara in his Marīchi Bhaṣya. Munīs'wara also noted the flaw but argues away in an untenable way. Another commentator named Ganes'a who was the author of the commentary named S'iromaṇipracās'a, does not seem to have noticed the flaw, or even if he did notice, probably he fought shy of pronouncing that there was a flaw. In fact a simple flaw like this in numerical examples, is not in the least derogatory to the prestige of Bhāskara. So, the commentators who happened to notice the flaw need not have pointed the same.

(4) Or again in the given place, for the value of H cos δ to be valid the Unnatakāla x must be such that

15 H sin $x > 13$ R so that H cos δ might be less than R. This means sin $x > \frac{13}{15} = .8667$ so that $x > 60°-4'$; so instead of 10 nādikas, if the time were given to be just even one Vināḍika greater, it would have been alright, or again if the latitude were given to be just a little less it would have been alright.

Verse 92. Oh! Mathematician! please tell me the magnitudes of the equinoctial shadow and the longitude of the Sun if at a place on a particular day, Kujyā is 245 and Taddhṛti 3125.

Verse 93. Answer to the questiou above.

$$s = \sqrt{\frac{144 \text{ Kujyā}}{\text{Taddhṛti}-\text{Kujyā}}} \text{ and H sid } \delta = \frac{12 \text{ Kujyā}}{s}$$

$$\text{and H sin } \lambda = \frac{\text{R H sin } \delta}{\text{H sin } \omega}.$$

Comm. From the fifth latitudinal triangle compared with third,

$$\underset{(1)}{\frac{\text{Kujyā}}{\text{Krāntijyā}}} = \underset{(2)}{\frac{\text{Krāntijyā}}{\text{Taddhṛti}-\text{Kujyā}}} = \underset{(3)}{\frac{\text{Agrā}}{\text{S. S.}}} = \underset{(4)}{\frac{s}{12}}$$

Multiplying (1) by (2) $\frac{\text{Kujyā}}{\text{Taddhṛti}-\text{Kujyā}} = \frac{s^2}{12^2}$

$\therefore \quad s = \sqrt{\frac{144 \text{ Kujyā}}{\text{Taddhṛti}-\text{Kujyā}}}$ Also Equating (1) and (4)

$$\text{Krāntijyā} = \frac{12}{s} \text{ Kujyā}.$$

Verse 94. Given that H sin δ + S. S. + Taddhṛti—Kujyā = 6720, and Kujyā+Agrā+H sin δ = 1960. Then I shall consider him who finds s and the longitude of the Sun as the very Sun illuminating the lotuses of astronomers.

Verse 95. Answer to the question above.

Divide 12 × Second sum by the first sum, that will be s. Again $\frac{12 \times \text{Second sum}}{12 + s + k} = \text{H} \sin \delta$. From H sin δ, λ could be had as before.

Comm. Comparing the third and fifth latitudinal triangles

$$\frac{}{\text{Krāntijyā}} = \frac{\text{Krāntijyā}}{\text{Taddhṛti} - \text{Kujyā}} = \frac{\text{Agrā}}{\text{S. S.}} = \frac{s}{12}$$

$$\frac{\text{Kujyā} + \text{Krāntijyā} + \text{Agrā}}{\text{Krāntijyā} + \text{Taddhṛti} + \text{S. S.} - \text{Kujyā}} = \frac{1960}{6720} = \frac{7}{24}$$

$$\therefore \quad s = 7/24 \times 12 = 7/2 = 3\tfrac{1}{2}''.$$

Again comparing the third and the first latitudinal triangles

$$\underset{(1)}{\frac{s}{\text{Kujyā}}} = \underset{(2)}{\frac{12}{\text{Krāntijyā}}} = \underset{(3)}{\frac{k}{\text{Agrā}}} = \underset{(4)}{\frac{s + 12 + k}{\text{Kujyā} + \text{Agrā} + \text{Krāntijyā}}} = \underset{(5)}{\frac{s + 12 + k}{1960}}$$

Equating (2) and (5) Krāntijyā $= \frac{12 \times 1960}{12 + s + k}$

$$\frac{12 \times 1960}{\frac{24}{2} + \frac{25}{2}} = \frac{12 \times 1960}{28} = 840 \text{ since when } s = 7/2$$

$k = 25/2$ which is the hypotenuse of the triangle formed by the equinoctial shadow with the gnomon.

Equating (1) and (5) Kujyā = 245 ; equating (3) and (5) Agrā = 875.

Now from the fourth latitudinal triangle compared with the first $\underset{(1)}{\frac{\text{Agrā}}{s}} = \underset{(2)}{\frac{\text{S. S.}}{12}} = \underset{(3)}{\frac{\text{Taddhṛti}}{k}}$

From (1) and (2) $\text{S. S.} = \frac{12}{7/2} \times \text{Agrā} = \frac{24}{7} \times 875 = 3000$

From (1) and (3) $\text{Taddhṛti} = \frac{k}{s} \times \text{Agrā}$

$$= \frac{25}{9} \times \frac{2}{7} \times 875 = 3125.$$

Note. This is a beautiful example exhibiting Bhāskara's dexterity in algebra.

Verse 96. Given that the sum of H sin δ, S. S. and Taddhṛti – Kujyā = 1440, and the sum of Agrā, S. S. and Taddhṛti = 800, I shall deem him whoever finds s and the longitude of the Sun, as the very Sun illuminating the lotuses of astronomers.

Verse 97. Answer to the problem above.

The second sum divided by the first and multiplied by 12 gives k from which s could be got. Then the first sum divided by $s + 12 + k$ gives H sin δ from which the longitude of the Sun could be got.

Comm. Comparing the third and the fifth latitudinal triangles, we have

$$\underset{(1)}{\frac{\text{Agrā}}{\text{Krāntijyā}}} = \underset{(2)}{\frac{\text{S. S.}}{\text{Taddhṛti} - \text{Kujyā}}} = \underset{(3)}{\frac{\text{Taddhṛti}}{\text{S. S.}}}$$

$$= \underset{(4)}{\frac{\quad}{12}} = \underset{(5)}{\frac{+ \text{S. S.} + \text{Taddhṛti}}{\text{Krāntijyā} + \text{Taddhṛti} - \text{Kujyā} + \text{S. S.}}} = \frac{\quad}{1440} = \underset{(6)}{\frac{\quad}{4}}$$

Equating (4) and (6) $k = \frac{12 \times 5}{4} = 15$

$\therefore k^2 = 225 = 12^2 + s^2 \quad \therefore s = 9$. Again comparing the fourth latitudinal triangle, with the fundamental,

$$\frac{\text{Agrā}}{s} = \frac{\text{S. S.}}{12} = \frac{\text{Taddhṛti}}{k} = \frac{\text{Agrā} + \text{S. S.} +}{s + 12 + k}$$

(1) (2) (3)

$$= \frac{1800}{9 + 12 + 15} = \frac{1800}{36} = 50 \qquad \text{II}$$

(4)

Equating (1) and (4) Agrā = $9 \times 50 = 450$

Equating (2) and (4) S. S. = $12 \times 50 = 600$

Thirdly Taddhṛti = $15 \times 50 = 750$

Again Equating (1) and (6) of I

$$\frac{\text{Agrā}}{\text{Krāntijyā}} = \frac{5}{4} = \frac{450}{\text{Krāntijyā}} \quad \therefore \quad \text{H} \sin \delta = \frac{450 \times 4}{5}$$

= 360 from which λ could be computed.

Verse 98. The chara at a place where $s = 9$, is equal to 3 nādis. If you could compute the longitude of the Sun, then certainly you are a leader among astronomers, Oh ! Scholar !

Verse 99. Answer to the problem above.

$$\frac{12 \text{ Carajyā}}{\sqrt{\left(\frac{12 \times \text{Carajyā}}{\text{R}}\right)^2 + s^2}} = \text{H} \sin \delta$$ where from λ the longitude of the Sun could be computed.

Comm. Let H sin $\delta = x$; then from the third latitudinal triangle $\frac{\text{Kujyā}}{\text{Krāntijyā}} = \frac{s}{12} = \frac{9}{12} = \frac{3}{4}$

$\therefore$ Kujyā = $\frac{3x}{4}$ since Krāntijyā means H sin δ

$$\therefore \quad \text{Carajyā} = \frac{3x}{4} \times \frac{\text{R}}{\text{H} \cos \delta} = \frac{3\text{R}x}{4\sqrt{\text{R}^2 - x^2}}$$

$= \text{H} \sin (3 \times 6) = \text{H} \sin 18^\circ \quad \therefore$ Squaring

$$9\,\text{R}^2 x^2 = 16\,(\text{R}^2 - x^2)\,\text{H} \sin^2 18 = 16\,\text{Carajyā}^2\,(\text{R}^2 - x^2)$$

$\therefore \quad x^2 \,(9\,R^2 + 16\text{ Carajyā}^2) = 16\,R^2\text{ Carajyā}^2$

$\therefore \quad x^2 = \dfrac{16\,R^2\text{ Carajya}^2}{9\,R^2 + 16\text{ Carajyā}^2} \qquad \therefore\; x = \dfrac{4\,R\text{ Carajyā}}{\sqrt{9\,R^2 + 16\text{ Carajyā}^2}}$

$= \quad 12\text{ Carajyā} \quad = \dfrac{12\text{ Carajyā}}{\sqrt{s^2 + \left(\dfrac{12\text{ Carajyā}}{R}\right)^2}}$

Here Carajyā being known, H sin δ could be computed.

Verse 100. If you studied what is known as Madhyamāharaṇa, then compute λ the longitude of the Sun given that H sin δ + H cos δ + H sin λ = 5000.

Verse 101. Answer to the problem above.

Let the given sum multiplied by 4 and divided by 15 be Ādya ; then H sin δ =

$$\text{Ādya} - \sqrt{910678 - \frac{2\text{ square of the given sum}}{337}}.$$

Comm. Let H sin $\delta = x$; then H cos δ =

$$\text{H sin } \delta = \frac{\text{H sin } \omega \text{ H sin } \lambda}{R}$$

$$= \frac{xR}{\text{H sin } \omega} = \frac{xR}{1397}$$

$\therefore$ The given sum $= x + \sqrt{R^2 - x^2} + \dfrac{xR}{1397} = 5000$

$$\overline{-x^2} = 5000 - x\left(1 + \frac{R}{1397}\right) = 5000 -$$

$\therefore \quad R^2 - x^2 = 5000^2 + x^2\left(\dfrac{4835}{1397}\right) - \dfrac{2 \times 5000 \times 4835}{1397}\,x$

$\therefore \quad x^2\left\{1 + \dfrac{4835^2}{1397^2}\right\} - \dfrac{2 \times 5000 \times 4835}{1397} = R^2 - 5000^2$

ie. $x^2\,(1397^2+4835^2) - 2\times 4835\times 1397\times 5000\,x = 1397^2\,(R^2 - 5000^2)$

ie. $x^2\,(25328834) - 2 \times 4835 \times 1397 \times 5000\,x = 1397^2\,(R^2 - 5000^2)$

$$\therefore\quad x^2 - \frac{2 \times 4835 \times 1397 \times 5000\,x}{25328834} = \frac{1397^2\,(R^2 - 5000^2)}{25328834}$$

Converting $\frac{[illegible]}{2835}$ into a continued fraction we have

$\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3} = \frac{4}{15}$ so that the equation could be written as

$$x^2 - 2 \times 5000\,\frac{\times 4}{15} = \frac{1397^2\,(R^2 - 5000^2)}{25328834}$$

Here $\frac{5000 \times 4}{15}$ is symbolized as Ādya so that we have

$x^2 - 2$ Ādya $x =$

$$5000^2 \times \frac{16}{225} - \frac{5000^2 \times 1397^2}{2532883} + \frac{1397^2 \times 3438^2}{2532883}$$

$$\left(\frac{16}{225} \quad \frac{1397^2}{2532883}\right) + \frac{1397^2 \times 3438^2}{2532883}$$

Here $\frac{16}{225} - \frac{1397^2}{2532883}$ is approximated to $\frac{-2}{337}$

and $\frac{1397^2 \times 3438^2}{2532883}$ is approximated to 910678 so that we have

$x =$ Ādya $\pm \sqrt{910678 - \frac{2\,s^2}{337}}$ where s is the given sum.

Since the positive sign of the radical is invalid because H sin $\delta \not> R$, so the negative sign is taken.

Verse 102. In a place where $s = 5''$, the sum of H sin δ, S. S., Taddhṛti, Kujyā and Agrā is 6500; find them individually *oh*, mathematician, if thou art adept in understanding the sphere and dealing with the latitudinal triangles.

Verse 103. Answer to the problem above.

Assuming H sin δ to be equal to 12 s and computing the various quantities cited; take their sum. Then by rule of three "If for this sum got, the individual magnitudes are such and such what will they be for the given sum" each can be had.

Comm. The cited magnitudes are respectively H sin δ, $\frac{\text{R H sin } \delta}{\text{H sin } \phi}$, $\frac{\text{R}^2 \text{ H sin } \delta}{\text{H sin } \phi \text{ H sin } \phi}$, $\frac{\text{H sin } \delta \text{ H sin } \phi}{\text{H sin } \varphi}$ and $\frac{\text{R H sin } \delta}{\text{H cos } \phi}$ which are all proportional to H sin δ, ϕ being given through 's'. With this idea of proportionality at the back of his mind, Bhāskara sets this ingenious question, and gives an easy way of solving it by assuming H sin δ to be $5 \times 12 = 60$, so that the others can be got rationally.

With this H sin δ, S. S. $\frac{3438 \times 60}{3438 \times 5/13} = 156$, Taddhṛti

$$\frac{3438^2 \times 60}{3438 \times \frac{5}{13} \times 3438 \times \frac{12}{13}} = 169 \text{; Kujyā} = \frac{60 \times 5}{13 \times 12/13} = 25$$

$$\frac{\times 60}{3438 \times 12/13} = 65$$

The sum of these is 475. So, by the rule of three mentioned above, H sin δ = 1200, S. S. = 3120. Taddhṛti = 3380, Kujyā = 500 and Agrā = 1300.

Or alternatively given $s = 5$, $k = 13$ so that H sin ϕ =

$$\not{b} = \frac{3438 \times 12}{13}.$$

the various magnitudes are H sin δ, $\frac{\text{H sin } \delta \times 13}{5}$,

$\frac{\text{H sin } \delta \times 13^2}{60}$, $\frac{\text{H sin } \delta \times 5}{12}$ and $\frac{\text{H sin } \delta \times 13}{12}$.

The sum of these is H sin δ $\left(1 + \frac{13}{5} + \frac{169}{60} + \frac{5}{12} + \frac{13}{12}\right)$

$$\text{H sin } \delta\left(\frac{60 + 156 + 169 + 25 + 65}{60}\right) = \frac{475}{60} \text{ H sin } \delta = \frac{95}{12}$$

H sin δ = 9500 ∴ H sin δ = 1200 from which by substitution the remaining magnitudes could be obtained.

Verse 104. If the sum of Agrā, H sin δ and Kujyā be 2000 find them individually, oh ! mathematician if thou be an adept in the geometry of the sphere and computation.

Comm. Here the quantities are respectively

$\frac{\text{R H sin } \delta}{\text{H cos } \varphi}$, H sin δ and $\frac{\text{H sin } \delta \text{ H sin } \varphi}{\text{H cos } \varphi}$ so that their

sum is H sin $\delta\left(1 + \frac{\text{R}}{\text{H cos } \varphi} + \frac{\text{H sin } \varphi}{\text{H cos } \varphi}\right)$

$$= \frac{\text{H sin } \delta \text{ (H sin } \varphi + \text{H cos } \varphi + \text{R)}}{\text{H cos } \varphi}$$

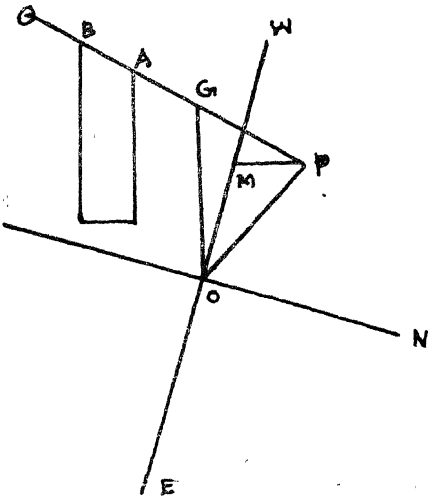
Here also we are to presume $s = 5$ so that the above sum is $\frac{\text{H sin } \delta\,(s + 12 + k)}{12}$ (by proportion of the first and second latitudinal triangles) = $\frac{\text{H sin } \delta\,(5 + 12 + 13)}{12}$ =

$\frac{5}{2}$ H sin δ = 2000 ∴ H sin δ = 800. Substituting this value in the above formula, Agrā

$$= \frac{\text{RH sin } \delta}{\cos \varphi} = \frac{3438 \times 800 \times 13}{3438 \times 12} = \frac{10400}{12} = 866\text{-}40\,;$$

$$\text{Kujyā} = \frac{\text{H sin } \delta \text{ H sin } \varphi}{\text{H cos } \varphi} = \frac{800 \times 5}{12} = \frac{4000}{12}$$

OG = Gnomon; OP = Shadow of the gnomon; PM = the Bhuja drawn from the extremity of the shadow P perpendicular on the East-west line; OM = Koti of the shadow extending along the East-west line. AB is the Nalaka placed along the Chayakarṇa PG. The eye is placed at A and the planet ⊙ is visible through the tube of the Nalaka AB.



Verses 105, 106 and 107. The method of observing through the instrument called Nalaka, the planetary position.

On a horizontal plane mark a point and through it draw the East-west line and also the North-south; if the planet is in the East mark off the computed Koti of the shadow towards on the East-west line; if the planet is in the Western hemisphere, mark this Koti towards the East. From the extremity of the Koti mark the computed Bhuja perpendicular to the East-west line and draw the computed shadow from the point so as to form a right-angled

triangle with the Bhuja and Koti. Extend a thread from the point of intersection of the bhuja and shadow to meet the gnomon's top so as to form the Chāyākarṇa or the hypotenuse of the right-angled triangle of which the other sides are the gnomon and the shadow. Along this thread place the Nalaka such that the lower extremity of the Nalaka coincides with the eye. Seeing through the Nalaka, the planet is to be seen. I shall tell how the planet could be seen in water as well.

Comm. The Nalaka is a simple tube formed generally of bamboo. The purpose of this is to verify the correctness of the computation of the shadow and its bhuja. If the computation is wrong the planet will not be seen in that direction. It might be asked how the shadow and bhuja are pertinent with respect to a planet, whose shadow cannot be observed as that of the Sun. True, but the computation of the shadow and bhuja are done as will be done with respect to the Sun, knowing the declination etc. as in the case of the Sun. Computation does not depend on the observation of the actual shadow. Computing the magnitudes of the Bhuja and Koti, the direction of the Chāyākarṇa points to the planet in the sky.

Verse 108. Observing the planet through the Nalaka in water.

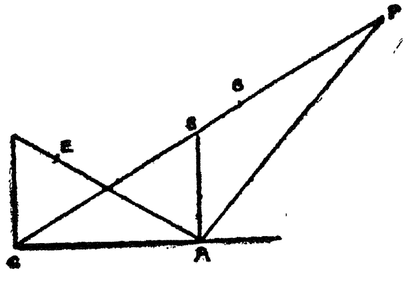


Fig. 63

Place the S'anku at the point of intersection of the Bhuja and shadow and holding the Nalaka along the join of the top of S'anku and the point, the planet could be seen in a basin of water placed at the point.

Comm. Let P be the planet casting the shadow AC of the gnomon AB. C the extremity of the shadow is the point of intersection of the shadow and the Bhuja. Though we have shown the gnomon in the position AB, it need not have been placed there in as much as we have the computed magnitudes of the shadow, the Bhuja and the Koti. Now we are directed to place the S'anku actually at C the point of intersection of the shadow and the Bhuja. Thus CD is the S'anku. Since CD=AB and and both are vertical evidently $\triangle s$ DBA and DCB are congruent. Hence $D\hat{C}B = D\hat{A}B$. But $D\hat{C}B$ = zenith-distance of the planet and as such is equal to $B\hat{A}D$ (also the zenith-distance of the planet) $\therefore$ $D\hat{A}B = B\hat{A}P$. Hence if a tray of water is placed at A, the planet will be visible as seen through DE, the Nalaka since the angle DAB is the angle of incidence and $B\hat{A}P$ the angle of reflection are equal.

Verse 109. The planet is to be shown to the king, who has an eye of appreciation for the same, either directly (as shown in fig. 62) in the sky or through water as shown in the fig. 63, having finished the preliminaries indicated.

Comm. Clear.

End of the Tripras'nādhyāya.

PARVASAMBHAVĀDHIKĀRA

Investigation into the occurence of an eclipse

Verses 1–2. Multiply the number of years that have elapsed from the beginning of the Kaliyuga by twelve and add the number of months elapsed from the beginning of the luni-solar year. Let the result be x. Then add

65 to x. Let the result by y. Then the longitude of what is called Sapāta-Sūrya or the longitude of the Sun with respect to a node will be x Raśis +

$\frac{(2y + 503)\,(1 + \frac{1}{169})}{3 \times 30}$ Raśis. If this longitude be less than 14°, then a lunar eclipse is likely to occur.

Comm. The first operation indicated above in directing x to be added to $\frac{2x\,(1 - \frac{1}{898}}{65}$ is intended to obtain the lunations that have elapsed from the beginning of the Kaliyuga. In this behalf we are asked to multiply the elapsed years by twelve to get the number of solar months. Here there is one subtlety to be noticed. The years that have elapsed are not entirely solar. In fact the years reckoned according to the luni-solar system were all originally luni-solar; but according to the convention of intercalary months, they were rendered solar upto the point of the latest intercalation, for, solar months plus intercalary months are equal to the elapsed lunations. From the moment of the end of the latest intercalary month, the subsequent years or year or fraction thereof would be luni-solar only. Nonetheless, no difference will be there in the computed Adhikamāsas in adding a few lunar months to the solar and taking them all to be solar. The maximum error committed in so doing will be of the order of (no. of days in a solar month minus no.

of days in a lunar month) multiplied by $36 \times \frac{2}{65} \times \frac{1}{30}$ of an adhikamāsa, assuming that an adhikamāsa would occur at the latest in 36 solar months. (In fact, an adhikamāsa would occur on the average in $32\frac{1}{2}$ solar months, but we have taken 36 roughly as the maximum figure in as much as the occurence of the Adhikamāsa might be belated on account of the convention stipulated). Thus the error would be $36 \times 2 \times \frac{2}{65} \times \frac{1}{30} = \frac{1}{13}$th of an adhikamāsa at the maximum. Hence, we are directed not only to construe that all the years elapsed to be solar but also the subsequent lunations of the current luni-solar year also to be solar months. Thus getting the number of elapsed months from the beginning of the Kaliyuga, the computation of the Adhikamāsas is formulated as follows. If in the course of 51840000 solar months of the Yuga there be 1593300 Adhikamāsas then during the elapsed solar months x, what is the number of elapsed Adhikamāsas? The result is

$$\frac{x \times 1593300}{51840000} = \frac{\dfrac{x \times 1593300}{796650}}{\dfrac{51840000}{796650}}$$

$= \dfrac{2 \times x}{65\text{-}4\text{-}21}$. Since Bhāskara knows that there will be two Adhikamāsas roughly in 65 solar months, he performed the above operation. This shows that for every 65 solar months roughly there occur two Adhikamāsas or more accurately a little less than two Adhikamās. So, taking, in the first instance 2/65 as the ratio of Adhikamāsas to the number of solar months, Bhāskara tries to find as to what quantity is to be subtracted from 2. That is found as follows. If there be A adhikamāsas in s solar months what will be the number of Adhikamāsas in x solar months? The result is $\dfrac{A\,x}{s}$. Again if there be two Adhikamāsas roughly in 65 solar months, how many will be there in x solar months? The answer is $\dfrac{2\,x}{65}$. But we have seen about that the

accurate number should be $\frac{2x}{65} - \lambda$ ie. a little less than $\frac{2x}{65}$. The question is now to find the value of λ. So,

equating $\frac{2x}{65} - \lambda$ to $\frac{Ax}{s}$, $\lambda = \frac{2x}{65} - \frac{Ax}{s} = x\left(\frac{2}{65} - \frac{A}{s}\right)$

$= x\left(\frac{2s - 65A}{65s}\right)$. Substituting for $2s - 65A$ namely

$2 \times 51840000 - 65 \times 1593300 = 115500$

$$\lambda = \frac{x \times 115500}{65 \times 51840000} = \frac{x \times 2 \times 57750}{65 \times 51840000} = \frac{2x}{65} \times \frac{1}{\frac{51840000}{57750}}$$

$$= \frac{2x}{65 \times 898} \quad \therefore \quad \frac{Ax}{s} = \frac{2x}{65} - \lambda = \frac{2x}{65} - \frac{2x}{65 \times 898}$$

$= \frac{2x}{65}\left(1 - \frac{1}{898}\right)$ as given.

The procedure, adopted as above, is in a way a short cut in Hindu Astronomy to obtaining a convenient convergent to a continued fraction. Let us use the method of continued fractions; the number of Adhikamāsas in x solar months is $\frac{Ax}{s}$ ie. $x \times A/s =$

$\frac{x \times 1593300}{51840000} = \frac{x \times 5311}{172800}$. Converting $\frac{172800}{5311}$ into a continued fraction we have

$32 + \frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\frac{1}{2} + \frac{1}{1+}\frac{1}{1+}\frac{1}{18+}\frac{1}{4}$ to which $\frac{65}{2}$ is a convergent but a good convergent is $\frac{245}{69}$. As this good convergent is unwieldy, Bhāskara used 2/65 and made amends for the roughness introduced by adopting it. Wherever a convenient convergent is not available, an easy and rough convergent is used and amends will be made for the rough-

ness resulting as follows. Let $\frac{M}{N}$ be a fraction to which $\frac{m}{n}$ is a convergent having small numbers as numerator and denominator, so that $\frac{M}{N}$ is taken to be equal to $\frac{m}{n}\left(1+\frac{1}{\lambda}\right)$.

Thus $\frac{M}{N}=\frac{m}{n}\left(1+\frac{1}{\lambda}\right)$ or $Mn = Nm\left(1+\frac{1}{\lambda}\right)$

$\therefore \quad Mn - Nm = \frac{Nm}{\lambda}$ or $\lambda = \frac{Nm}{Mn - Nm}$. In the present case M is the number of Adhikamāsas, and N the number of solar months. $\frac{m}{n}$ if taken to be $\frac{2}{65}$

$$\therefore \quad \text{In this case } \lambda = \frac{-2 \times \text{Solar months}}{65 \times \text{Adikamāsas} - 2 \times \text{Solar months}}$$

which is indicated in the commentary by Bhāskara.

In this context, it may be mentioned that a Karaṇagrantha named Nārasimha based upon Sūryasiddhānta (A Karaṇagrantha is a manual according which the Hindu calendar is computed with easy numbers without undergoing the laborious process indicated in the treatises called Siddhāntas like the present Siddhānta S'iromani. In these Karaṇagranthas, instead of taking the beginning of the Kalpa or Mahāyuga or the Yuga, as the epoch, a recent date ie. the date of the author of the Karaṇagrantha is taken as the epoch, and processes using approximations are adopted for the sake of ease. Naturally therefore these Karaṇas (as they are also called) get easily obsolete within the course of a few hnudreds of years so that a fresh Karaṇa is called for preparation, if the calculations were to accord with the Siddhāntas which those Karaṇas profess to follow. In fact, the present Karaṇa of Nārasimha written in 1333 S'aka year ie. in 1411 A.D. declares that a previous Karaṇa named Tithicakra reported to have been written by one Mallikārjuna Suri grew obsoleto and

so, a fresh Karaṇa is being written to accord with the original Sūrya Siddhānta, (Vide verses I, II of Nārasimha) "तिथिचक्रं यत् प्रणीतं मल्लिकार्जुनसूरिणा, कालेन महता तस्मिन् खिलीभूते तदादरात्, नौपुरी सिङ्गार्यस्य नरसिंहेन सूनुना, एतदेव स्फुटतरं क्रियते सौरसम्मतम्" This Karaṇa takes 3/98 instead of 2/65 as a convergent in computing the Adhikamāsas. In this case the continued fraction becomes $32 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}$. So that the convergents are $\frac{32}{1}$, $\frac{33}{1}$, $\frac{65}{2}$, $\frac{98}{3}$. Taking 98/3 as a convergent, let us see how it was sought to make amends for the roughness of the convergent. As per the Sūrya-siddhānta the number of Adhikamāsas during the course of 51840000 solar months of a Yuga are 1593336 as against 1593300 prescribed by Bhāskara. Hence putting

$$\frac{1593336}{51840000} = \frac{3}{98}\left(1 + \frac{1}{\lambda}\right)$$

ie. $\frac{66389}{2160000} = \frac{3}{98}\left(1 + \frac{1}{\lambda}\right)$, $\lambda = \frac{3 \times 2160000}{66389 \times 98 - 3 \times 2160000}$ $= 248$ very approximately. So Nārasimha adopts for the Adhikamāsas instead of $\frac{2}{65}\left(1 - \frac{1}{898}\right)$, the formula 3/98

Having added the Adhikamāsas so obtained to the solar months we have the lunations that have elapsed from the beginning of the Kaliyuga. Then the next procedure indicated is as follows. In an eclipse solar or lunar, the celestial latitude of the Moon has to be less than a particular limit for the occurence of an eclipse. Thus in the case of a solar eclipse the celestial latitude of the Moon must be less than 32′, whereas in the case of a lunar eclipse it is to be less than 56′. Of course, for a solar eclipse to be possible anywhere on the earth, not for a given place, the limit is far higher as given in texts of modern astronomy namely $p_m + s + m - p_s$ where p_m is the horizontal parallax of the Moon, p_s that of the

and s and m the angular semidiameters of the Sun and the Moon. (This formula we shall see later). This higher limit comes to 88.5′. The limit of 56′ for the occurence of a lunar eclipse is the value of

$$p_m + p_s - s + m \text{ as we shall see later.}$$

The latitude of 56′ of the Moon arises out of a longitude of 12° of the Moon with respect to a node, whereas the latitude of 32′ arises out of a longitude of 7° with respect to the node. Since at an eclipse solar or lunar, the longitude of the Moon with respect to a node, is the same as the longitude of the Sun with respect to the same or opposite node, the latter must be 12° for the occurence of a lunar eclipse. But as the difference between the mean and true Suns is about 2°, the longitude is stipulated as 14°. In other words, for the oecurence of a lunar eclipse, the longitude of the Sun on the full-Moon day with respect to the nearer node shall be less than 14°. To compute this longitude of the Sun with respect to the nearer node on a full-Moon day, we are given the subsequent procedure indicated in the verse. In 53433300000 lunations of the Kalpa, the sum of the sidereal revolutions of the Sun and the Node (Rāhu) (Sum because Rāhu has a retrograde motion) is equal to 455231168 which is equal to 455231168 × 12 = 54627734016 Raśis. Then in one lunation what will be the increase of the longitude with respect to the Node? The result is

$$\frac{54627734016}{53433300000} = 1 \text{ Raśi} + \frac{3583302048°}{5343330000}\left(= \frac{74652126}{111319375}\right)$$

dividing by 48 both the numerator and denominator. Taking the first two digits in the numerator and denominator of the fraction the fraction is approximately equal to $\frac{36}{54}$ or 2/3. Taking this as a convergent we make amends for the roughness as follows.

$$\frac{74652126}{111319375} = \frac{2}{3}\left(1 + \frac{1}{A}\right) \therefore A = \frac{2 \times 111319375}{3 \times 74652126 - 2 \times 111319375}$$

$= \frac{222638750}{1317628} = 169$ approximately. Hence the increase of the Sun's longitude with respect to a node is

1 Rāśi $+ \frac{2}{3}\left(1 + \frac{1}{169}\right)^{\circ}$ I. In the beginning of the Kaliyuga, the longitude of the node was 5 Rāśis–3°–13′ and the arc moved by the Sun with respect to the node during the course of half a lunation is 0—15—20, so that their sum is 5 –18 –33. Here we have added for half a lunation because the context is a lunar eclipse and the beginning of the Kaliyuga was a New Moon day. Also, at the beginning of the Kali, the Mean Sun being at the zero-point of the zodiac, the negative longitude of the node only is the longitude of the Sun with respect to the node. Hence we have to add the above longitude of 5 –18 –33 to the longitude obtained through the above formulation I, which means 168°—33′ is to be added to $\frac{2x}{3}\left(1 + \frac{1}{169}\right)$ where x is the elapsed number of lunations. Taking 168°—33′ as nearly equal to 168°—40′, $\frac{2x}{3}\left(1 + \frac{1}{169}\right) + 168\frac{2}{3}^{\circ}$

$$= \frac{2x}{3}\left(1 + \frac{1}{169}\right) + \frac{506}{3} = \frac{2x}{3}\left(1 + \frac{1}{169}\right) + \frac{503}{3}\left(1 + \frac{1}{169}\right)$$

approximately $= \frac{(2x + 503)}{3}(1 + 1/169)$ as formulated.

Thus for x lunations, the longitude of the Sun with respect to the node is x Rāśis $+ \frac{(2x + 503)}{3}\left(1 + \frac{1}{169}\right)^{\circ}$. If this longitude falls short of 14°, we could except a lunar eclipse.

Latter half of verse 3 and *verses* 4, 5. Particularity with respect to a solar eclipse.

Add half a Rāśi to the longitude previously obtained; find out on which side the Sun lies, north or south; compute the longitude of the Sun from the number of days

elapsed after the Samkramaṇa day (ie. the day on which the Sun has left one Rās'i and entered another); obtain the hour-angle in nādis of the Sun at the ending moment of the Amāvāsyā ie. at the moment of New Moon; add or subtract one-fourth thereof in Rās'is from the position of the Sun according as the Sun is in the Western or Eastern hemisphere; then finding the declination of that point and from the sum or difference of the declination and latitude of the place, obtain the zenith-distance of the culminating point of the ecliptic; taking that point to be roughly the Vitribha ie. the point of the ecliptic which is 90° behind the Sun on the ecliptic, find one-sixth of the zenith-distance; taking the sum or difference of the result and the longitude of the Sun with respect to the node (got in the beginning by adding half a Rās'i to his position at full-Moon) if the result happens to fall short of 7°, then we could expect a solar eclipse.

If there be no eclipse at the current New Moon, then go on adding 1 Rās'i $-0° -40' -15''$ to the longitude of the Sun with respect to the Node (which will be his longitude for the moment of the next New Moon) and repeating the procedure indicated, the occurence of an eclipse or otherwise could be known. If occurence be indicated then compute the actual positions of the Sun, Moon and Rāhu and following the procedure to be indicated in the chapter on solar eclipses, the moment of the occurence of the eclipse and other relevant details could be computed.

Comm. In the case of a lunar eclipse, the Manaikyārdha ie. half the sum of the diameters of the eclipsing and eclipsed bodies (namely the cross-section of the Earth's shadow at the lunar orbit and the Moon) is 56′. This is the maximum limit to the celestial latitude of the Moon if an eclipse were to occur, and this latitude will be there if the longitude of the Moon with respect to the nearer node is 12°. Since a lunar eclipse occurs at the moment of a full Moon, the distance of the Sun then from the opposite node

should be also 12° for the occurrence of an eclipse. Since it is customary to check the occurence of an eclipse through Sapātasūrya ie. longitude of the Sun with respect to the node (the prefix Sa is to signify that the sum of the Sun's longitude and that of the node should be taken, as the longitude of a node is measured in the opposite direction from the zero-point of the ecliptic) it is stipulated that the Sapātasūrya should be 12° for the occurence of a lunar eclipse. But, as the True Sun might differ from the Mean by about 2°, and as we are concerned with the True Sun only, the limit is increased by 2°, so that a lunar eclipse may occur if the Sapātasūrya happens to be less than 14°. Thus there is no more complication with respect to the occurence of a lunar eclipse than requiring the longitude of the Sapātasūrya to be less than 14° for the occurence of a lunar eclipse. If this condition be satisfied, there will be an eclipse and that will be visible at all places, where there is night, since a body in shadow will not be seen from any place whatsoever.

But, there is a complication with respect to the occurence of a solar eclipse namely that it is not a question of the Sun entering a shadow. The Sun could never be shadowed. A solar eclipse occurs when the disc. of the Moon comes in between the Sun and an observer and obstructs a vision of the Sun's disc. The Moon being very near us compared with the Sun, its coming in between the Sun and an observer may well be compared with a cloud obstructing the vision of the Sun. Just as, when a cloud obstructs the vision of the Sun for an observer, it could not do so with respect to another who is situated at a distance, so also, if the Moon effects a solar eclipse for a particular place, it could not do so for all places. This is said to be due to parallax or Lambana as it is called (Refer fig. 64). It is so called because, when there is an eclipse of the Sun for an imaginary observer at the centre C of the Earth, the Moon intersecting in the line of sight to

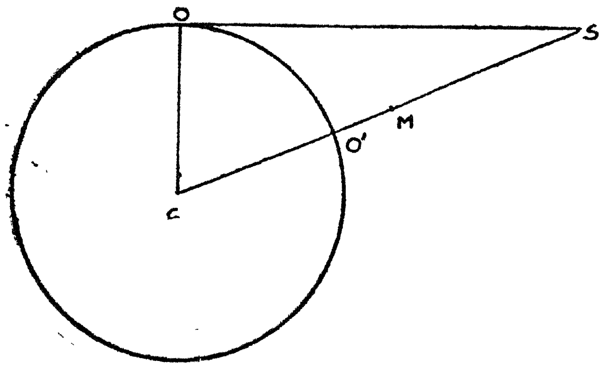


Fig. 64

the Sun, for an observer 'O' on the surface of the Earth, the Moon is not in the line of sight namely OS but hangs down that line (लम्बते अनेनेति लम्बनम् = That phenomenon by which the Moon hangs down the line of sight). Hence it is not sufficient to say that the Sapātasūrya has a longitude of 7° to conclude the occurence of a solar eclipse for a place. If the Sapātasūrya be less than 7°, certainly there will be a solar eclipse for some place on the earth but not for all places. So, to decide whether there will be a solar eclipse for a given place, we are to take into account the phenomenon of parallax. At every New Moon, the longitudes of the Sun and the Moon will be no doubt equal; yet the Moon may not obstruct a vision of the Sun, not being situated in the ecliptic plane. He may be above the ecliptic plane or below it and if he be within 32′ from the plane, a part of the Moon's globe may hide a part of the Sun's from the vision of certain observers who are situated about the point o' of fig. 64. But suppose an observer is at O. For him. there is no eclipse at all as could be seen from the figure, As the observer moves away and away from O′ towards O, the effect of parallax will be greater and greater in longitude, whereas, as the observer moves away and away from O′ towards O_1 (where O_1 is the geocentric pole of the circle

(c) shown in the figure) along the circle of intersection of the Earth with a plane through CO′ perpendicular to the plane of the paper, where O_1 is a point on the earth such that $S\hat{C}O = 90°$ the effect of parallax will be greater and greater in latitude. In other words the parallax has both an effect in longitude as well as in latitude. When it has an effect in longitude only it is called Lambana, whereas, when it has an effect in latitude, it is called Nati. (Thus the translation of 'parallax' as Lambana alone is not fully correct, though at times the parallax may have its complete effect in longitude only or in latitude only). For the observer who moves in the ecliptic plane only as the one moving from O′ towards O, the parallax will have its entire effect in longitude only and for the observer moving in the perpendicular plane from O′ to O_1 mentioned before, the parallax will have its entire effect in latitude only. For observes other than the two above, it will have effect both in longitude and latitude also. When parallax effects longitude the time of conjunction is preponed or postponed, whereas when it effects latitude, the latitude of the Moon appears to have increased or decreased. When it increases, no eclipse occurs and when it decreases an eclipse does occur. At O′, the latitude will be exactly what has been computed; at the point of intersection O″ of the join of the centres of the Sun and Moon with the surface of the Earth, parallax nullifies the latitude and on the great circle O′ O″ there will be parallax in latitude only effecting the magnitute of the latitude.

It will be seen that for the point O″, the Sun and the Moon are in the zenith, so that neither will suffer from parallax. For the point O the Sun will be on the horizon and the Moon being depressed below the horizon though he is in geocentric conjunction the parallax in longitude or lambana is maximum and the occurence of the New Moon had already elapsed 4 nādis ago.

it will be seen that at the points O_1 and O_2 where O, is the other geocentric pole of the circle (c) drawn there cannot be an eclipse, the latitude being incresaed (as per Hindu astronomy) by 48′ – 46″. In fact, there will be eclipse for the places on either side of O″ (the point of intersection of the join of the centres of the Sun and Moon with the Earth's surface) to such a distance as will increase the latitude to 32′ only. For the other places on O′ O″ beyond these points, the latitude of the Moon exceeds this limit and so there will be no eclipse. Further clarification of Lambana and Nati will be given later.

The above analysis underlies our investigation for the occurence of a solar eclipse. Sapātasūrya might be less than 7°, but it does not mean that every place will enjoy an eclipse. So, for the place concerned, we have to see that, even after taking into account the parallax in latitude ie. in Nati, still the latitude will be less than 32′. To obtain this parallax in latitude, the method adopted is to find it at the point called Vithribha, (nonagesimal) ie. the point which is behind the Lagna the rising point of the Ecliptic by 90°; for, as we shall see in the Chapter on Solar eclipses, the parallax in latitude at the Vitribha will be equal to the parallax in latitude at any point of the ecliptic. In other words, wherever be the Sun and Moon on the ecliptic (Moon also being very near the node may be taken roughly to lie on the ecliptic) to compute the amount by which the latitude is increased, we compute it for the Vitribha, and this will hold good for the arbitrary position of the Moon, for, there also the latitude will be increased by the same amount. The procedure given in verse 4 is to locate the Vitribha from the position of the Sun and to find its zenith-distance to compute the Nati; or rather, it is to locate the culminating point and taking it roughly to be Vitribha, to compute the influence of Nati on the latitude or S'ara. If it were only to find the Vitribha, it could be computed from the

lagna of the moment of New Moon. Computation of the zenith-distance of the Vitribha is a little cumbrous, so that, for brevity, it is sought to compute the culminating point, obtain its declination and thereby its zenith-distance which could be taken to be the zenith-distance of the Vitribha also, from which the Nati is calculated.

We are directed to obtain first the Nata or the hour angle of the Sun for the moment of New Moon. We know on that particular New Moon day how long Amāvāsya will last after Sun-rise ie. we know when the actual moment of New Moon occurs on that day. We also know the duration of day time on that day so that subtracting the time of occurence of the New Moon after Sun-rise from half the duration of day, we obtain the hour angle of the Sun (Nata) in nādis. Now the Moon's longitude is effected by parallax, the effect being depression of the Moon. It is roughly estimated that the hour angle expressed in nādis is increased by $\frac{1}{4}$ of its value on account of this. Strictly speaking the effect of parallax is far more on the position of the Moon than on the Sun. But the Hindu procedure apparently treats the Sun alone for parallax. The reason is that at the moment of geocentric conjunction of the Sun and the Moon, when we consider the combined effect of parallax on the Moon and the Sun at once, for a given place, we may as well compute the relative position of the Sun effected by parallax. Let the hour-angle of the Sun in nādis be x. Then effected hour-angle will be $x\left(1 + \frac{1}{4}\right) = \frac{5x}{4}$ nādis. But each Rās'i being taken roughly to rise in 5 nādis, the hour-angle in Rās'is will be $5x/4 \div 5$ Rās'is $x/4$. Hence we are directed to divide the hour-angle of the Sun in nādis to divide by 4. This $x/4$ being substracted from the longitude of the Sun, we get the longitude of the culminating point. Then we are directed to obtain the declination of the culminating point from the formula $H \sin \delta = H \sin \omega \times H \sin \lambda \div R$,

This declination of the culminating point being known, and the latitude being known, its meridian zenith-distance could be got. Take that to be roughly the zenith-distance of the Vitribha. Then an approximate estimate of the effect in latitude is obtained as follows. Let the zenith-distance of the Vitribha be x. Then if we have for H sin z = R, the maximum effect of 48′ – 46″ in the latitude, what shall we have for H sin 45° ? The result is

$$\frac{\text{H sin } 45 \times 48' - 46''}{3438} = \frac{2431}{3438} \times 48' - 46'' = 34' - 30''.$$

Then again another approximate estimate is made as follows. Let the Sapātasūrya (Sapātasūrya = longitude of the Sun or what is the same of the Moon with respect to the Node) be λ. Then for $\lambda = 15°$, we have a latitude of 70′. That being so, for a variation of 34′ – 30″ in the latitude, what will be the corresponding variation in the longitude λ ?

The result is $= \frac{69}{2} \times \frac{15}{70} = \frac{207}{28} = 7° \frac{11}{28}$ which is roughly one-sixth of 45°. Hence if the zenith-distance of the culminating point taken to be the Vitribha be 45°' the variation in the Sapāta-sūrya (= Sapāta chandra) will be one-sixth thereof. Hence we are asked to increase or decrease as the may be, the Sapāta sūrya by 1/6th. If the resulting Sapātasūrya be less than 7°. there may be an eclipse. When the lunar orbit lies north of the ecliptic, the culminating point of ecliptic being south of the zenith, the latitude of the Moon is decreased by parallax, so that, we have to decrease the Sapātasūrya; whereas when the lunar orbit is then south of the ecliptic, the effect of parallax is to increase the latitude and consequently, we have to increase the Sapātasūrya. Or again when the culminating point of the ecliptic is north of the zenith and the lunar orbit is north of the ecliptic, parallax appears to increase so that we have to increase the Sapātasūrya;

when at that moment the lunar orbit is south of

the ecliptic, parallax appears to decrease the latitude so that we have to decrease the Sapātasūrya.

The proportion that 70′ of latitude correspond to 15° of Sapāta-chandra is due to the formula

$\frac{H \sin 15^\circ \times H \sin 4\frac{1}{2}}{R} = H \sin \beta$. Using logarithms,

log sin $\beta = 9{\cdot}4130 + 8{\cdot}8946 = 8{\cdot}3076$ so that $\beta = 1^\circ - 10' = 70'$. This proportion could be used because the Moon is within 15° of the Node. Hence when we are investigating the occurence of a solar eclipse, application of rule of three is not unjustified or crude.

Herein, Bhāskara assumed the zenith-distance of the nonagesimal to be round about 45° and drew a conclusion that the effect in the longitude on account of parallax in latitude is one-sixth of the zenith-distance. Instead if the zenith-distance be assumed to be z, the result would be $\frac{48' - 43'' \times \sin z \times 15}{70} = \frac{21}{2} \sin z$ which may be taken as a better approximation. If, on the other hand the modern value of 57' of the lunar parallax be taken, the result would be $\frac{57 \sin z \times 15}{70} = 12 \sin z$ approximately which is a better value.

In this calculation, it is better to compute the zenith-distance of the nonagesinal using modern methods instead of assuming the nonagesimal to be on the meridian.

LUNAR ECLIPSES

Verse 1. The ritualistic purpose served at the time of an eclipse.

Scholars of Smṛtis and purāṇas declare that prayer, charity or offerings to gods made in fire at the moment of an eclipse conduce to much spirituality. Hence I give hereunder the methods of computing the moment of an eclipse (lunar or solar) in as much as such a knowledge apart from its religious importance, is also wrought with a beautiful mathematical treatment.

Verse 2. The initial procedure to be adopted to compute an eclipse.

To know the occurence of a solar eclipse, find the exact moment of the New Moon, which is indicated by the equality of longitudes of the Sun and the Moon, and to know the occurence of a lunar eclipse compute the exact moment of the full Moon which is indicated by the fact that $M = S + 180°$ where M and S are the longitudes of the Moon and the Sun, agreeing in degrees, minutes and seconds, though differing in Rasis by six. Also compute the longitude of the Node (Rāhu) for the moment as directed.

Comm. Having ascertained the possibility for the occurence of a solar eclipse, we are directed to compute the positions of the Sun, the Moon and Rāhu for the day of the New Moon. The Sun and the Moon are to be rectified for corrections like Desāntara, Bhujāntara, Udayāntara etc. From the elongation of the Moon the tithi is to be computed and the method of successive approximations called Chālana Karma is to be used to obtain the exact moment of conjunction. This process of

Chālana is as follows. At first knowing the elongation of the Moon at Sunrise, by rule of three, using the then daily motions of the Sun and Moon, the moment of the conjunction is to be computed. Again the positions of the Sun and Moon are to be computed for that moment and also their daily motions are to be rectified for the moment. With these rectified daily motions and with the then positions of the Sun and the Moon, again the moment of conjunction is to be computed. Repeating the process till an invariable answer is reached, we have the exact moment of conjunction. For that moment, the position of the Rāhu is also to be calculated. Similar is the procedure for a lunar eclipse also. It is to be noted that the correction called Natakarma, which we formerly identified to be the correction for 'Astronomical Refraction' is also prescribed here as particularly mentioned by Bhāskara, in the commentary. (Vide verses 68, 69 Spaṣṭādhikāra).

Verse 3. The magnitndes of the orbits and the orbital radii of the Sun and Moon.

The distances of the centres of the globes of the Sun and the Moon from the centre of the Earth in yojanas are respectively 689377 and 51566.

Comm. We saw in the Kakshādhyāya of the first chapter as to how these distances were estimated. Some scholars pronounced these distances are parameters; but as per the modern estimate of the Earth's radius as compared with that of Bhāskara, (The method indicated by him in the Bhuparidhimānādhyāya of chapter I, is quite mathematical) a yojana equals five modern miles and with this correspondence, the distance of the Moon is very near the truth. So to say that the distances given above are mere parameters is wrong; also if one of the parameters gives a correct value of the quan-

tity in question, the other also should; but because the latter does not, to call them parameters is merely meaningless.

It is interssting to note that in the commentary under this verse, Bhāskara says "If for a circumference of 3927, the diameter will be 1250, ..." This means that Bhāskara takes $\pi = \frac{3927}{1250} = 3{\cdot}1416$ which compares very well with the modern value 3·14159. The value of π adopted by Bhāskara in this, seems to have been taken from Lallācharya's Siṣyadhīvṛddhida, Chandragrahaṇādhikāra verse no. 3.

"शरयमाङ्गहता 625 भनवाप्तिहृत्
ग्रहवृतिश्रवण: फलमुच्यते"

Verse 4. Computation of what is called the 'Kalākarṇa'.

The radius vector is to be computed even in the case of the Equation of centre as we did in the case of S'īghraphala. If it be 'K', $\frac{R^2}{2R - K}$ will be what is Kalākarna both in the case of the Sun, as well as the Moon.

Comm. While obtaining the Equation of centre, the formula used was $\frac{r \sin m}{R}$ whereas, strictly speaking, it should have been as in the case of the S'īghraphala, after effecting the so called 'Karṇānupāta' $\frac{r}{K}$ sinm. While trying to answer why this Karṇānupāta was not done there also, Brahmagupta gave such an answer as made Bhāskara exclaim 'यतो विचित्रा फलवासनाऽत्र' i.e. 'It is really curious in this respect.' Bhāskara was really a most rational type of astronomer, and one will not fail to appreciate his sense of rationality when he declares that (1) "अस्मिन् णितस्कन्धे उपपत्तिमानेव आगम: प्रमाणम्"; and when he was

unable to adduce a proof he declares (2) 'उपलब्धिरेव वासना' meaning thereby (1) 'In this branch of science, we reckon only such an authority which has a proof behind it ie. which could be substantiated' and (2) 'There is no proof in this but accordance with observations alone has to be taken as a proof'.

Though in the case of formulating the equation of centre Karṇānupāta was not stipulated to simplify matters, as there was not much difference, the Equation of centre being generally small. It is to answer such contexts as this that Bhāskara said in the Goladhyāya

' ie. 'If we in some particular context do not mention certain things it should not be condemned because in such contexts, (1) there is not much difference or (2) no useful purpose is served to a good extent or (3) it is too clear as does not require to be mentioned (4) the procedure implies a lot of cumbrous calculations and the result is after all negligible and (5) Mention will make the work on hand too unwieldy and brevity which is the soul of wit is to be sacrificed. In the present context this procedure of 'rectification of the Karṇa' is sought to improve matters. Bhāskara's words 'यदा ग्रहस्य कर्णं उत्पन्नः तदा कर्णो
ः' seem really to imply that in the formula

$\frac{r}{R}$ H sin m for the equation of centre in the place of R,

we are called upon to substitute really K. Though we had been in default for not doing so in the context of the Equation of centre, there is no reason why we should not make up the deficiency in this context. So, therefore, this rectification of Karṇa is stipulated here.

The procedure originally called for a rectification is that taking K to be R, we have to compute r and again taking the resulting K to be R, we have to compute r and so on repeating the process till an invariable value for K is obtained. This means that we should go on substituting for r, $\frac{r\,K}{R}$. Instead of following this laborious process of 'Asakṛt-Karma' ie. method of successive approximations, Bhāskara gives an alternative in the verse, whieh is as follows.

Let K be the value of the Karṇa, for a value r of r. Since we are directed to make this K as R, ie. we have to add R—K to K thus making it R, we add also R—K to R, to keep the relative position of R and K to be almost the same. In other words considering the fraction $\frac{K}{R}$, adding R−K to both the numerator and denominator we have $\frac{R}{2\,R-K}$ whieh means that for a radius 2 R −K, the Karṇa will be R; that being so for a Radius R what will be the Karṇa ? The result is $\frac{R^2}{2\,R-K}$ as given.

It will be noted that the above interpolative procedure is adopted as a short cut technique to the otherwise laborious process. The mathematical correctness of this procedure will be seen from the following analysis.

The problem is to change the Mandaparidhi to a radius K of the deferent by the formula (as indicated by Bhāskara in the course of the commentary) $r' = \frac{r\,K}{R}$ so that $\delta r = r^1 - r = \frac{r\,K}{R} - r = \frac{r\,(K-R)}{R}$ (1) Now we have $K^2 = R^2 + r^2 + 2\,R\,r \cos m$ construing R and m as constants we have to find δ K for $\delta\,r$

Differentiating $2\,K\,\delta K = 2\,r\,\delta r + 2\,R \cos m\,\delta r$

ie. $\delta K = \dfrac{\delta r\,(r + R\cos m)}{K}$. But from fig. 65 (which is a portion of the epicyclic figure)

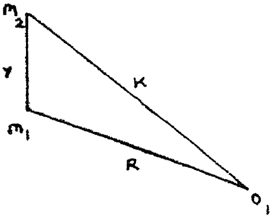


Fig. 65

$\hat{M}_1 = 180 - m$

$\hat{O}_1 = \theta$ = Mandaphala

$\hat{M}_2 = m - \theta$

$r = K\cos\overline{m-\theta} - R\cos m$ so that $r + R\cos m = K$ as $(m-\theta)$. Substituting in the above,

$$\delta K = \frac{\delta r \times K \cos\overline{m-\theta}}{K} = \delta r \cos\overline{m-\theta}$$

But δr from (1) is $\dfrac{r\,(K-R)}{R}$

$\therefore\ \delta K = \dfrac{r\,(K-R)}{R}\cos\overline{m-\theta}$. But from the triangle of fig. 65. $K = R\cos\theta + r\cos\overline{m-\theta}$ so that $r\cos\overline{m-\theta} = K - R\cos\theta$. But θ being small $\cos\theta$ may be taken to be unity so that $r\cos\overline{m-\theta} = K - R$. Again substituting in the above

$\delta K = \dfrac{(K-R)^2}{R}$ (2) Now as per the formulation of Bhāskara

$$K^1 = \frac{R^2}{2\,R-K} = \frac{R^2}{R+R-K} = \frac{R}{1+\frac{R-K}{R}} = \frac{R}{1-\frac{(K-R)}{R}}$$

$$= R\left(1 - \frac{K-R}{R}\right)^{-1}$$

$\left|\dfrac{K-R}{R}\right| < 1$, expanding binomially,

$$K^1 = R\left(1 + \frac{K-R}{R} + \frac{(K-R)^2}{R^2}\right) = R + K - R + \frac{(K-R)^2}{R}$$

$= K + \frac{(K-R)^2}{R}$ $\therefore$ $K^1 - K = \delta K = \frac{(K-R)^2}{R}$ as found above.

Note. Bhāskara, having formulated this, appeals to 'Dhulīkarma' ie. arithmetical computation, for convincing those who may not be able to follow his logic. Here one may note also the wrong directive given by the Sams'odhaka in the text. The proof furnished by us above gives a mathematical veracity to Bhāskara's formulation.

Verse 5. To rectify the Yōjanakarṇa or the spatial radius Vector.

The above Kalākarṇa multiplied by the Karṇa given in Yōjanas and divided by the Radius gives the rectified Yōjanakarṇa.

Comm. In the formula given above $\delta K = \frac{(K-R)^2}{R}$ which is in units of spatial minutes (on the scale of R = 3438).

where K′ is the rectified Kalā-karṇa, and y the Yōjanakarṇa given in verse 3, gives the rectified Yōjanakarṇa.

Second half of verse 5. The spherical radii of the Sun and the Moon.

The spherical diameters of the Sun and the Moon are respectively 6522 and 480 Yojanas.

Comm. The word 'Bimba' is used to connote the spherical diameter. The diameter of Moon as given will be equal to 480×5=2400 miles in modern terms which is not far from truth. Once we accept that the method indicated by us in the Kakshādhyāya of chapter I was that

followed by the ancient Hindu Astronomers to estimate the distance of the Moon, the spherical radius the horizontal parallax, the orbital radius pertaining to the Moon could all be deduced and the magnitudes so deduced accord with their average values in modern astronomy. • The magnitudes pertaining to the Sun however, should be deemed as parameters.

Verse 6. $e - \frac{(S - E)\, K_m}{K_s} = 2\alpha$ where e = Earth's diameter, s = The Sun's diameter; Km = Moon's distance from the Earth's centre; Ks = The Sun's distance from the Earth's centre, and α = radius of the Earth's shadow cone at the lunar orbit.

Comm. In fig. 66, let CD be the radius of the Earth's shadow cone at the lunar orbit. Required to find the magnitude of CD. Triangles DEF and ESG are similar

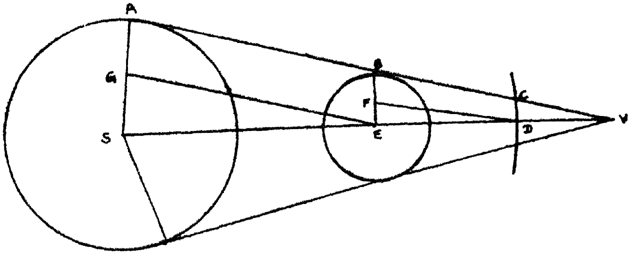


Fig. 66

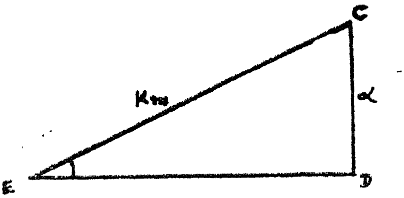


Fig. 67

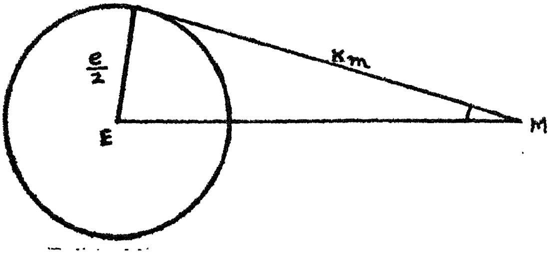
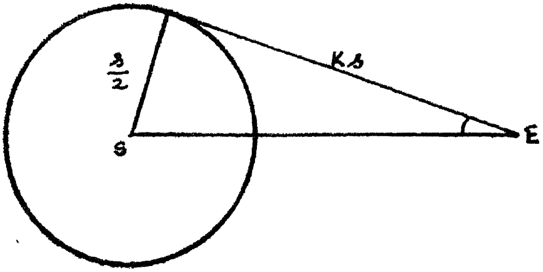


Fig. 68



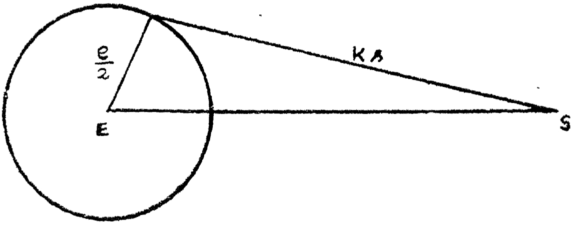


Fig. 70

where DF and EG are drawn parallel to VBA, the common tangent.

$$\therefore \quad \frac{EF}{DE} = \frac{SG}{ES} \quad \text{ie.} \quad \frac{\frac{1}{2}e - a}{Km} = \frac{\frac{1}{2}s - \frac{1}{2}e}{Ks}$$

where a = FB = CD required

$$\therefore \quad a = \tfrac{1}{2}\left\{e - \frac{K_m}{K_s}(s-e)\right\} \quad \text{I}$$ as formulated $2a$ being what is called Rāhu-Bimba or diameter of the Earth's shadow cone at the lunar orbit which is called Ku-bhā Vistṛti in the verse (Ku = Earth ; Bha = Shadow ; Vistṛti = diameter).

Note (1) We shall prove that this formula accords with the modern formula given for the radius of the shadow cone. Divide I throughout by 2 K_m, so that

$$\frac{a}{K_m} = e/2K_m - \frac{s-e)}{2\,K_s} \quad \text{II}$$

But from fig. 67. $a/K_m = \sin \hat{E} = \hat{E}$ = angular radius of the shadows cone expressed in radius. From fig. 68, $\frac{e}{2\,K_m}$ = Horizontal parallax of the Moon ; $\frac{s}{2\,K_s} = \sin \hat{E}$ $= \hat{E}$ = angular radius of the Sun expressed in radians from fig. 69 ; and $e/2K_s$ = (from fig. 70) Horizontal parallax of the Sun. Thus Equation II means $\rho = P - \sigma + P^1$ III where ρ = angular radius of the shadow cone, P = Horizontal parallax of the Moon ; P^1 = that of the Sun and σ = angular radius of the Sun.

Note (2) If we don't divide I by 2 K_m, we have the radius of the shadow-cone in Yojanas, substituting the values of e and s, K_m and K_s.

Note (3) It is worth hearing Bhāskara in his commentary under this verse. Observe the Sun's disc while rising on the day when his true motion is equal to his mean, with a compass composed of two rods hinged at one end and carrying a protractor at the other. We get the mean diameter of the Sun equal to 32′ - 31″ - 33‴. Similarly, observe the Moon's disc on a full-moon-day

when his true motion equals the mean. It will be 32′ - 0″ - 9‴.

Note (4) Substituting the values of P, P^1 and σ in III $\rho = 52'\text{-}42'' + 3'\text{-}57'' - 16'\text{-}16'' = 40'\text{-}23''$ = Angular radius of the shadow-cone at the lunar orbit which almost accords with the modern value 41′–49″.

Note (5) One may wonder as to how, taking wrong values for s and Ks, such a correct value could be obtained for P. In equation II, α, K_m, e are all near the truth so that the terms effected are s/K and e/Ks; but both s and Ks being parameters s/Ks comes off alright, which is the angular radius of the Sun's disc which could be measured. The only vitiating term is e/Ks which is the horizontal parallax of the Sun which was overestimated unwittingly by a wrong supposition as indicated in the Kakshādhyāya. However e/Ks comes to be 3′–57″ and this overestimate is mitigated to some extent that the Earth has an atmosphere which boosts the angular radius of the shadow cone by about 1′ and the remainder of the overestimate makes amends for the smaller value of P taken.

Verse 7. To convert spatial measures into angular measure. The diameters of the Sun, the Moon, and Rāhu in Yōjanas multiplied by R = 3438′, and divided respectively by Ks, K_m and K_m give their angular measures.

Comm. From fig. 69, $\frac{s/2}{Ks} = \sin \hat{E} = \hat{E}$ expressed in radians $= \frac{E'}{3438}$ $\therefore$ $E' = \frac{3438 \times s}{2\,Ks}$ which means that the angular radius of the Sun is got by multiplying s/2 ie. the spherical radius of the Sun by R = 3438 and dividing by Ks as mentioned. Similar is the case with respect to the other two.

Note. The word Kalākaraṇa is used to signify 'To convert into angular measure.

Verse 8. An alternative method of obtaining the angular radii.

The daily motion of the Sun increased by one-tenth of its value and halved, gives the angular diameter of the Sun. The Moon's daily motion multiplied by 3 and divided by 71, gives the angular diameter of the Moon. Or the daily motion of the Moon being decreased by 715 and divided by 25 and the result being added to 29 gives the angular diameter of the Moon.

Comm. This method gives in an easy way the true angular radii. The formulae given are $s' = \frac{1}{2}s_1\,(1+\frac{1}{10})$; and $m' = \frac{3\,m_1}{74} = \frac{m_1 - 715}{25} + 29$. This may be elucidated as follows. The argument used is "If the spherical diameter of 6522 Yojanas corresponds to a spatial daily motion of $11858\frac{3}{4}$ Yojanas, what angular diameter corresponds to the angular daily motion s_1?". The proportionality is clear and the result is

$\frac{s_1 \times 6522}{11858\frac{3}{4}} = \frac{26088}{47435}$. Converting $\frac{26088}{47435}$, into a continued fraction, it is

$\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{94}$. The penultimate convergent is $11/20 = \frac{1}{2}\,(1+\frac{1}{10})$. The formula follows.

Similar calculation gives m'.

Note. The advantage of these formulae is that they are not only easy but also adopting the true daily motion we have the true angular radii. This procedure was adopted by Bhāskara from Brahmagupta. The latter, however, prescribes a nearer convergent namely 10/247 but actually $\frac{17}{420}$ is the nearest convergent.

The next formula namely $m' = \frac{m_1 - 715}{29} + 29$ is approximate. This may be elucidated as follows. Let the

daily motion be 715 ; then as per the previous formula the angular diameter should be $\frac{3}{74} \times 715 = \frac{2145}{74} = 28\frac{73}{74} = 29'$ very approximately. The mean daily motion is 790 which corresponds to 32′ of angular diameter. Taking advantage of this arithmetical correlation namely that the excess of 3′ over 29′ corresponds to 75′ of daily motion. Bhāskara gives the formula $m' = \frac{m_1 - 715}{25} + 29$. This formula correctly holds good when $m_1 = 740$, for, equating $\frac{3x}{74} = \frac{x - 715}{25} + 29 = \frac{x + 10}{25}$, x will be equal to 740. For other values between 715 and above it holds very approximately. Thus, when $m_1 = 715$, $m' = 28\frac{73}{74}$ ie. 29 when $m_1 = 740$, $m' = 30$, when $m_1 = 765$, $m' = 31\frac{1}{75}$ (error $\frac{1}{75}$) when $m_1 = 790$, $m' = 32\frac{1}{37}$ (error $\frac{1}{37}$) and so on.

Verse 9. An alternative method of finding the angular diameter of the shadow cone.

$2\rho = 2/15\, m_1 - 5/12\, s$, where m_1 and s_1 are the daily motions of the Moon and the Sun respectively.

Comm. In the previous verse, we had formulae to compute the diameters of the discs of the Sun and Moon, knowing their daily motions. Since in practice we have these daily motions computed for every day, so the computation based upon those daily motions conduces to ease in the matter of calculation. Now in this verse, the radius of the shadow cone is also calculated in terms of the daily motions of the Sun and the Moon, which is more an ingenious device adopted in practice. The elucidation of the formula depends on the following technique as conceived by the Hindu astronomers. In as much as the Sun's sphere is far bigger than that of the Earth, the shadow of the Earth assumes the form of a cone. From a knowledge of the decrease in the diameter, as we proceed from the

Sun to the Earth and a knowledge of the Sun's distance, we can compute similar decrease as we proceed from the Earth to the lunar orbit knowing the distance of the Moon. Such a decrease measured in Yojanas is termed by Bhāskara as 'अपचययोजनानि' ie. Yojanas of decrease in diameter. It was this concept that led to the formulation of $2a$ in Yojanas in the form $2a = e - \frac{(s-e)\,K_m}{K_s}$ of verse 6 and is indeed based upon the similarity of triangles as proved by us in that context. Dividing the above equation by 2 K_m, we have

$$a/K_m = \frac{e}{2\,K_m} - \frac{1}{2}\frac{(s-e)}{K_s} \quad \text{I}$$

Dividing a, e and $(s-e)$ thus by the distances K_m and K_s is termed 'Kalā-Karaṇa' ie. converting spatial distance into angular measure. Thus dividing a by K_m is converting the radius of the shadow cone into angular measure at the lunar orbit; dividing $\frac{1}{2}e$ by K_m is estimating the angular measure of the earth's radius as seen from the Moon's distance or what is the same the horizontal parallax of the Moon, whereas dividing $\frac{1}{2}s$ by K_s is getting the angular radius of the Sun's disc as seen from the Earth and dividing $\frac{1}{2}e$ by K_s is getting the angular radius of the Earth's disc as seen from the Sun or what is the same the horizontal parallax of the Sun.

Equation I which gives a/K_m the angular radius of the shadow cone, may also be interpreted as follows. $\frac{e}{2K_m}$ = Horizontal parallax of the Moon as mentioned above which is equal to the angle (fig. 66) $\widehat{ECB}$, for, $\frac{1}{2}e$ = EB; K_m = EC so that $e/2K_m$ = EB/EC = sin $\widehat{ECB}$ = $\widehat{ECB}$ expressed in radian measure. Also

$$\frac{s-e}{\;} = \frac{SA-GA}{SE} = \frac{SG}{SE} = \sin \widehat{SEG} = \widehat{SEG} = \widehat{EVB}$$

(alternate angle) = Semivertical angle of the shadow cone (say θ). Now $\widehat{ECB} - \widehat{EVB} = \widehat{CED}$ = angular measure of CD ie. the angular radius of the shadow cone (expressed in radian measure).

Converting $\frac{1}{2}e/K_m$ into angular measure, the proportion used by Bhāskara is "If by the daily spatial motion of $11859\frac{3}{4}$ Yojanas of the Moon, we have its daily motion in arc, what shall we have for $e = 1581$ Yojanas?" The result is $1581/11859\frac{3}{4}\, m_1$. Converting the coefficient into a continued fraction we have $\frac{1}{7+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{175}$......... The penultimate convergent is $\frac{2}{15}$. Hence $e/K_m = \frac{2}{15}\, m_1$.

Converting $\frac{1}{2}\left(\frac{s-e}{Ks}\right)$ into arc, the proportion used is "If by the daily spatial motion of $11859\frac{3}{4}$ Yojanas, we have the daily motion of s_1, what shall we have for $s - e = 6522 - 1581 = 4941$ Yojanas?" The result is $\frac{4941}{11859\frac{3}{4}}\, s_1$

Converting the coefficient into a continued fraction, we have $\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{146}$. The penultimate convergent is $\frac{5}{12}$. Hence, the result is $\frac{5}{12}\, s_1$.

Note (1) It might be asked whether the Hindu astronomers used the theory of continued fractions. The answer is, they did though they did not write the continued form in the form we do now. They arranged the successive quotients in a vertical line and called the column as a 'Valli' or 'creeper'. One may refer to the chapter in Bhāskara's Bijagamita on 'Kuttaka' in this context.

Note (2) The formula derived above to obtain tne Rāhu-Bimba or diameter of the shadow-cone at the lunar orbit, is one which could be conveniently used in practice,

for, as mentioned before, the daily motions of the Moon and the Sun are ready computed for every day. Also the advantage in using this formula is that besides the fact that we need to deal only with small quantities instead of the big numbers of Yojanas, the true value for the day of eclipse is got by using the true daily motions. Using the mean daily motions, however, we have for the mean diameter, $\frac{2}{15} m_1 - \frac{5}{12} s_1 = \frac{2}{15} \times 790'\text{-}35'' - \frac{5}{12} \times 59'\text{-}8''$ $= 105\frac{2}{5} - 25\frac{2}{3} = 81$ very approximately.

Note (3) In obtaining a convergent for $\frac{s-e}{Ks}$, since the values of s and Ks are parameters s/Ks is got alright, where $\frac{e}{Ks}$ is more exaggerated as the value of the horizontal solar parallax. By this term the result is increased to an extent of 3′ out of which 1′ is mitigated by the fact that we have to take the earth's atmosphere also into consideration, for, that will increase the radius of the shadow to $\frac{1}{50}$th of its value.

Note (4) The last line of verse (9) is "The earth's shadow eclipses the Moon, and the Moon eclipses the Sun". This statement is deliberately made by Bhāskara to *remove the misconception in the minds of lay men* who wrongly believe the usage of the words राहुग्रस्त and केतुग्रस्त used in the calendars even today and the mythological puranic story associated with Rāhu and Kētu depicting that a serpent devours the Sun and the Moon. In fact the shadow of the earth which eclipses the Moon and the shadow of the Moon which hovers on the earth at the time of a solar eclipse do resemble the tail of a serpent.

Verse 10. To obtain the latitude of the Moon.

Vikṣepa or S'ara as it is also called ie. the latitude β of the Moon is obtained by the formula

$= \frac{\text{H} \sin \lambda \times 270}{\text{R}}$ and it will have the same direction as

the Moon with respect to the ecliptic where λ is the longitude of the Moon with respect to the nearer node and 270′ or $4\frac{1}{2}^\circ$ is taken to be the inclination of the lunar orbit to the ecliptic or what is the same, the maximum latitude of the Moon.

Comm. The formula is evident and similar to that for calculating the declination of the Sun. Thus

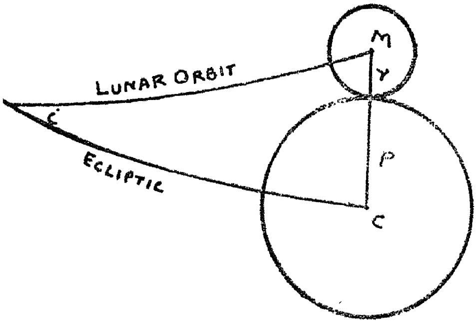
$H \sin \beta \quad \frac{H \sin \lambda \times H}{R} \quad \frac{1}{2}^\circ$ But since β is small and also $4\frac{1}{2}^\circ$, we can take $\beta = \frac{H \sin \lambda \times 270'}{R}$

The argument used by Bhāskara, however, is 'If by a H sin λ equal to R, we have the maximum latitude of 270′, what shall we have for H sin λ?' The result is as given.

Note. The modern value for i the inclination of the lunar orbit to the ecliptic is given to vary between 4°-58′

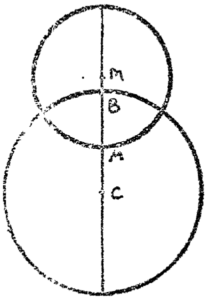
Verse 11. The definition of the magnitude of a lunar eclipse.

Sthagita or the magnitude of an eclipse is defined as $P+r+\beta$ where P and r are respectively the radii of the eclipsing and eclipsed bodies and β is the latitude of the Moon. If the Sthagita is greater than $2r$, then the eclipse is total.



Comm. In figure 71, the eclipsing body is just contacting the eclipsed body. Taking the case of a lunar eclipse, the latitude then of the Moon is evidently $P+r$ ie. $\beta=P+r$ holds good at the moment of first contact. (C) is the cross-section of the shadow-cone at the lunar orbit and (M) is the Moon. (fig. 72), $CB+AM-CM=CB+AB+BM-CM=AB+(CB+BM)-CM=AB+CM-CM=AB$

$\therefore$ $P+r-\beta=AB=$ Sthagita. Thus Sthagita gives the portion of the diameter of the eclipsed body which is shadowed. The eclipsed body is termed the Chādya, the eclipsing body as the Chādaka and $P+r$ as the Manaikya-ardha ie. half the sum of the diameters of the eclipsing and eclipsed bodies.



72

When the Sthagita exceeds the diameter of the eclipsed body the eclipse is evidently total ie., when

Verse 12. Duration of the eclipse and duration of its totality.

Sthiti-Khanda Duration of the eclipse

$$\text{Marda-Khanda} = \frac{\sqrt{(P-r)^2-\beta^2}\times 60}{m_1-s_1} = \tfrac{1}{2}\text{ Duration of totality}$$

where P is the radius of the shadow-cone, r the radius of the Moon's disc, β its latitude taken to be constant during the eclipse, m_1 and s_1 the daily motions of the Moon and Sun respectively.

Comm. (1) The time between the moment of first contact and the middle of the eclipse or the moment of

opposition or conjunction as the case may be is called *Spars'a-Sthiti-Khanda.*

(2) The time between the middle of the eclipse and the moment of last contact is called the *Mokṣa-Sthiti-Khanda.*

(3) The time between the commencement of total eclipse and the middle of the eclipse is called the *Sammīlana-Marda-Khanda.*

(4) The time between the middle of the eclipse and the end of total eclipse is called *Unmīlana-Marda-Khanda.*

The suffix Khanda meaning 'half' is generally omitted while refering to these phases. In the above verse we are given formulae for Sthiti-Khanda and Marda-Khanda only without specifying whether they pertain to Spars'a or Mokṣa. Though the same formulae serve for both the Spars'a phase as well as the Mokṣa phase under the supposition that β does not vary, it will be noted that the Spars'a-Sthiti-Khanda will not be equal to Moksha-Sthiti-Khanda and that the Spars'a-Manda-Khanda will not be equal to the Moksha-Marda Khanda in as much as β changes from moment to moment.

In fig. 73, let C_1 be the position of the eclipsing body at the moment of first contact and C_2 its position at the moment of last contact. In the figure is shown only one position of the Moon's disc signifying that we may consider the motion of the eclipsing body keeping the eclipsed body fixed (or what is the same relative to the position of the eclipsed body). It is evident from the fig. that $C_1M = C_2M = P+r$ so that $C_1M\,C_2$ is an Isosceles triangle. Let MN be the $\perp^{ar}$ dropped from M on $C_1\,C_2$. $C_1\,C_2$ is the ecliptic because the centre of the shadow will be moving along the ecliptic, for, in fig. 66, SE the ecliptic passes through D the centre of the cross-section of the shadow-cone, as well as through the vertex V of the shadow-cone.

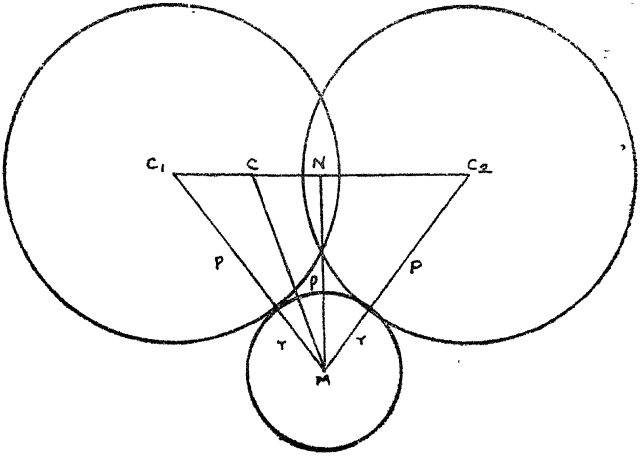


Fig. 73

MN is therefore the latitude of the Moon. Since the latitude is not the same at the moment of first contact and that of the last contact, the figure drawn does not represent the true figure but only a figure drawn on the supposition that β remains the same and C_1 Moves relative to M. From the figure $C_1N^2 = (P+r)^2 - \beta^2 =$

The Sthiti-Khanda defined in this verse is the time taken by C_1 to reach the position N ie. the position at the moment of opposition, and again from the position N to the position C_2. The velocity of C_1 relative to M is no other than the excess of the velocity of, the Moon over that of the Sun. (The velocity of the Earth is the relative velocity of the Sun with respect to the Earth and this is equal to the velocity of the shadow moving along the ecliptic). So, the time taken by C_1 to reach the position of N relative to the Moon is equal to $\frac{60 \times \sqrt{P + r^2}}{m_r -}$

Similarly the time taken by the centre of the shadow from N to C_2 ie. from the point of opposition to the moment

of last contact has also the same formula where in each case β is the latitude at the moment of opposition. The path taken by the centre of the shadow is called 'ग्राहकमार्ग' ie. the path of the eclipsing body. The actual case when both C and M are both moving and when β is considered as a non-changing quantity is shown in fig. 74. In this

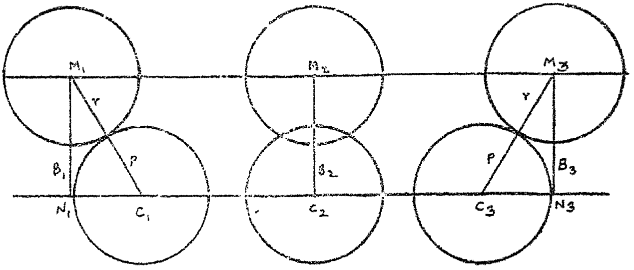


Fig. 74

case, three positions are shown, (1) that at the first contact (2) that at opposition and (3) that at last contact, where C_1, M_1, C_2, M_2 and C_3, M_3 give the positions of the centre of the shadow and that of the Moon's disc respectively, both the centres being shown as moving. Since the Moon moves faster than C and as such overtakes C, the path of M from M_1 M_3 which synchronizes with the path of C from C_1 to C_3, is shown to be longer. But, one may wonder, how C_1 N_1 and C_3 N_3 represent the Spars'a-Sthiti-Khanda and Mokṣa-Sthiti-Khanda respectively. The distance overtaken by M with respect to C from the point of first contact to the point ef opposition is $M_1 M_2 - C_1 C_2 = C_1 N_1$. Hence we compute $C_1 N_1$ by the formula $C_1 N_1^2 = (P+r)^2 - \beta^2$. Similarly from the point of opposition to the point of last contact M overtakes C by the distance $M_2 M_3 - C_2 C_3 = C_3 N_3 =$

Fig. 75 shows the situation when β changes as is the actuality. When the opposition takes place after the Moon crosses the node, then $\beta_3 > \beta_2 > \beta_1$, whereas if

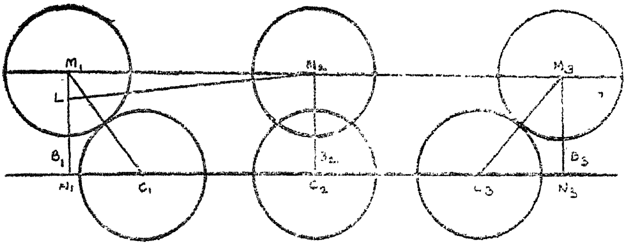


Fig. 75

the opposition precedes the Moon's position at the node $\beta_3 < \beta_2 < \beta_1$. Also, when β changes, $M_1 M_2$ does not exceed $C_1 C_2$ exactly by $C_1 N_1$. So, on both the counts, the formulae, given in verse 12 are approximate. What is done in practice is that β is computed for the moment of opposition and estimating the Spars'a-Sthiti-Khanda by the formula given above, and subtracting it from the time of opposition the moment of first contact is got. Then β is computed for that time and again the formula is applied to get the *Spars'a*-Sthiti-Khanda. Repeating the process, we rectify the Spars'a-Sthiti-Khanda. Even then, we do not have the actual value of the Spars'a-Sthiti-Khanda, because $M_1 M_2$ does not exceed $C_1 C_2$ exactly by $C_1 N_1$.

A more correct procedure would be to compute the time between the moment of first contact and the moment of opposition and by that time, to compute the length of

$M_1 M_2$ and take ·in the place of m' and use the

2

formula of verse 12. · This nicety, however, need not be attended to with respect to the duration of totality, for, it does not make much difference.

Another way of obtaining a better value for T, the Spars'a-Sthiti-Khanda is to take average values for β_1 and β_2, m_1 and m_2, s_1 and s_2 where m_1 and m_2 are the values of the Moon's daily motion and s_1 and s_2 are those of

Sun's at the point of first contact and the moment of conjunction respectively.

We may also use calculus to obtain δT, the variation in time for a variation of $\delta\beta$ in β and a variation of δm_1 in m_1 ignoring the small variation in s_1, as follows.

$$T^2 = \frac{(P+r)^2-\beta^2}{m_1-s_1}$$

$$\therefore \quad 2T\,\delta T = \frac{(m_1-s_1)\times -2\beta\,\delta\beta-\overline{P+r^2-\beta^2}\,\delta m_1}{(m_1-s_1)^2}$$

$$\therefore \quad \delta T = -\frac{\beta\delta\beta}{T\,(m_1-s_1)} - \frac{\delta m_1\,(\overline{P+r}^2-\beta^2)}{T\,(m_1-s_1)^2}$$

The first term on the Right hand side gives the variation for $\delta\beta$ and the second for δm_1.

Fig. 76 shows the case of totality.

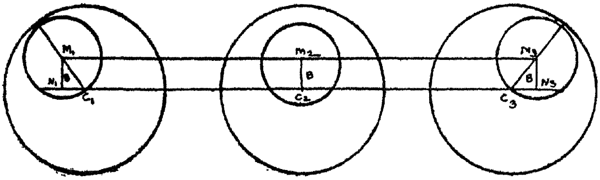


Fig. 76

$M_1 M_2 = N_1 C_1 + C_1 C_2$ $\therefore$ The Moon has to overtake C from the moment of the beginning of totality to the moment of opposition by the distance $C_1 N_1$ with a relative velocity of m_1-s_1. Hence the time of Sammīlana-Marda-Khanda is equal to

$$\frac{\sqrt{C_1M_1^2-\beta^2}\times 60}{m_1-s_1} = \frac{\sqrt{(R-r)^2-\beta^2}6\times 0}{m_1-s_1}$$

as given, taking β to be constant. Similarly the Unmīlana-Marda-Khanda from the position $(M_2\,C_2)$ to the position $(M_3\,C_3)$ will also be the same, taking β to be constant.

Rectification of this time, when β is considered as varying will proceed on the same lines as before.

Verse 13. Rectification of the times of Spars'a-Sthiti-Khanda and Mōksha-Sthiti-Khanda.

From the position of the Moon and that of the Node obtained for the moment of opposition, have to be computed their positions for the moment of first contact and those for the moment of last contact. For this, $\frac{T \times v}{60}$ is to be subtracted and added respectively to the positions at the moment of opposition of the Moon and Node, where T is the time of the Spars'a-Sthiti-Khanda, and v the daily motion (of the Moon or the Node as the case may be). From these positions β has to be computed for the moment of first contact and that of last contact, and from this β the time of Spars'a-Sthiti-Khanda and Mōksha-Sthiti-Khanda have to be rectified by the method of successive approximation.

Comm. From fig. 75, $C_1 N_1$ and $C_2 N_2$ are the distances gained by the Moon over the centre of the shadow so that to get their correct values β_1 and β_2 are to be used and not β. Hence β_1 and β_2 are to be computed using T the time of Spars'a-Sthiti-Khanda and that of the Mōksha-Sthiti-Khanda which are taken to be equal in the first instance. Since T is the time taken as a first approximation, β_1, β_2 are also approximate in the first instance. From these β_1, β_2 T is to be rectified and in this rectification, we have T_1 and T_2 differing, as the times of Spars'a-Sthiti-Khanda and Mōksha-Sthiti Khanda. From these rectified times again β_1, β_2 are further to be rectified and from them again T_1, T_2 are to be further rectified. This procedure is to be continued till constant values are obtained for T_1 and T_2.

Note. T_2 will be less than or greater than T_1 according as $\beta_2 \gtrless \beta_1$.

Verse 14. Rectification of the Sammīlana-Marda-Khanda and Unmīlana-Marda-Khanda.

Proceeding on the same lines as above and obtaining β_3 and β_4 the rectified latitudes of the Moon for the moments of the commencement and end of totality of the eclipse, the Sammīlana-Marda-Khanda and Unmīlana-Marda-Khanda, T_3 and T_4 are to be rectified.

Note. We have the formula $\sin\beta = \sin\lambda \sin i$ so that by differentiating. we have $\cos\beta\delta\beta = \cos\lambda\delta\lambda \times \sin i$

$$\therefore \quad \delta\beta = \frac{\sin i \cos\lambda\, A\lambda}{\cos\beta}$$

This formula gives in one stroke the rectified latitudes of the Moon at the respective moments from which the respective rectified times could be got.

Verse 15 *and the first half of verse* 16. The definition of Bhuja and the method of finding it at an intermediate point of time.

The word 'Iṣta' is used to connote 'At any given time'. The word 'Spārśika-Iṣta' means 'At a given time after the moment of first contact'; similarly the word 'Maukṣika Iṣta' means 'At a given time before the moment of last contact'. $(T-t)\ (m_1-s_1)$ where (m_1-s_1) is in degrees (m_1 and s_1 of course being expressed in minutes); T stands for the Sthiti-Khanda (Spārśika or Maukṣika) and t stands for the Iṣta (Spārśika or Maukṣika) gives the Bhuja. Similarly with respect to obtaining the Marda-Bhuja. (The former is called Sthiti-Bhuja).

Comm. In fig. 77, let C and M be the centres of the Rāhu (cross-section of the shadow-cone at the lunar orbit)

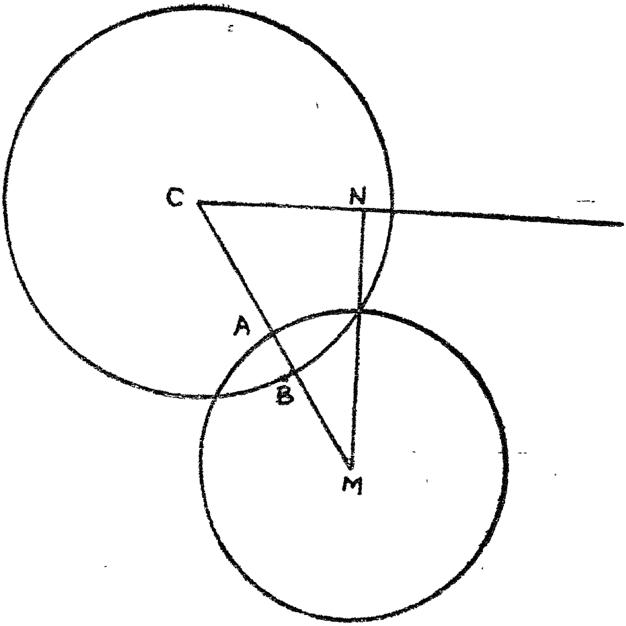


Fig. 77

and the Moon respectively; let MN be the perpendicular from M on the Grāhaka-mārga or the path of the eclipsing body (ie. the ecliptic). Then CN is called the Bhuja at the time.

At the moment of first contact, the value of CN is $\overline{\ ^2-\beta^2}$ where β is the latitude of the Moon at that moment. At any subsequent moment, from fig. 77, CN is equal to $\sqrt{(P+r-AB)^2-\beta^2}$ where β is the latitude at the subsequent moment and AB the portion of the radius of the eclipsed body shaded. Hence we could obtain the Bhuja at any subsequent moment, by computing the latitude at that moment and the value of AB. But AB could be computed only by knowledge of CN and β. Hence

the necessity for knowing the value of CN at any subsequent moment arises. β, of course, could be computed, knowing the hourly variation of β, which in its turn could be known, by a knowledge of the hourly variation in λ, the longitude of the Sapātachandra.

The magnitude of CN is calculated by the rule of three "If by the Spars'a-Sthiti-Khanda we have initially the initial value of CN, what shall we have for $(T-t)$?" The result is $\frac{(T-t) \times CN}{T}$ where CN is the initial value of CN and T the Spars'a-Sthiti-Khanda. Substituting the values of CN and T from verse 12, where

$$CN = \sqrt{(R+r)^2-\beta^2} \text{ and } T = \frac{\sqrt{(R+r)^2-\beta^2} \times 60}{m_1-s_1}$$

we have the required Bhuja as

$$\frac{(T-t)\,(\sqrt{R+r^2-\beta^2})}{60\,(\sqrt{R+r^2})-\beta^2)} \times (m_1-s_1) = \frac{(T-t)\,\{m_1-s_1\}}{60} \text{ minutes}$$

$= (T-t)\,(m_1-s_1)$ degrees as given.

Similarly we could find the Bhuja with respect to 'totality' ie. the 'Marda-Bhuja' as it is called.

Note. One might mistake M_1 M_2 of fig. 74 (M_1 pertaining to a subsequent moment) to be the Bhuja defined above, which is the join of the centre of the eclipsing body and the foot of the latitude at the middle of the eclipse. That is why Bhāskara uses the word 'Madhya-S'arāgra-Chihna' in the commentary, meaning thereby not the foot of the actual latitude at the middle of the eclipse but only the point N of fig. 77 which 'signifies' it.

Second half of verse 16 *and first half of verse* 17.

Taking the latitude of the Moon at a given time as Kōti, and Bhuja as the Bhuja of the moment defined above, we have the Karṇa of the moment as $\sqrt{\text{Bhuja}^2+\beta^2}$; $R+r-$Karṇa gives the Grāsa at the moment.

Comm. The word 'Grāsa' at the moment stands for AB of fig. 77, Karṇa for CM, where CN is the Bhuja and MN is the Kōti. The 'Grāsa' at the moment of opposition has the special name Sthagita.

Second half of verse 17 *and verse* 18.

To obtain the time after the moment of first contact, knowing 'Grāsa' at the moment.

$$T-\frac{\sqrt{(P+r-\text{Grāsa})^2-\beta^2}}{m_1-s_1}=t;$$ this 't' is to be rectified by obtaining the β of the moment and again finding t and repeating the process till an invariable magnitude is got.

Comm. This is the converse of finding the Grāsa given the time. The method of rectification is also evident. In the above equation considering β and t as variables, and differenting,

$$\delta t=\frac{1\times-\beta\delta\beta}{2\,(m_1-s_1)\sqrt{(P+r-g)^2-\beta^2}}=\frac{-\beta\delta\beta}{(T-t)\,(m_1-s_1)^2}$$

Knowing $\delta\beta$, δt could be got without taking recourse to the method of successive approximations.

Verse 19. Certain definitions.

The 'Middle of the eclipse' (or strictly speaking the moment when the portion eclipsed is a maximum) occurs at the moment of opposition. Spars'a or Pragraha is at the moment of first contact and Mokṣa is at the moment of last contact, separated from the moment of the middle of the eclipse by times equal to Spars'a-Sthiti-Khanda and Mokṣa-Sthiti-Khanda respectively before and after. Similarly Sammīlana and Unmīlana or the moment of the commencement of totality and the end thereof occur before and after the moment of 'the middle of the eclipse' by times equal to Sammīlana-Marda-Khanda and Unmīlana-Marda-Khanda respectively.

Comm. Clear.

Verse 20. To get what is called the Valana.

The hour-angle of the eclipsed body expressed in nādis, multiplied by 90 and divided by half the duration of night (if it be lunar eclipse) or half the duration of day (if it be solar) as the case may be will give the degrees of an angle, whose H sine being multiplied by the H sine of the latitude and divided by (H cos δ, (where δ is the declination of the eclipsed body), gives the H sine of what is called Ākṣavalana which is north when the hour angle is east, and south otherwise.

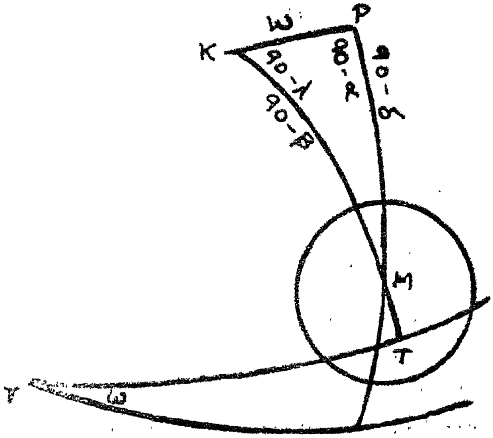
Comm. This subject of Valana requires a detailed treatment as is given in the Golādhyāya by Bhāskara. Here only a practical formula is given to proceed with the computation. For an understanding of this formula we have to necessarily draw upon the treatment in Golādhyāya.

The word 'Valana' means 'deflection'. The problem posed is at what point of the disc of the eclipsed body does the eclipse begin and at what points it ends. Since an observer sees the disc of the eclipsed body on the background of the spherical surface of the sky, the specification of the point of first contact must neeessarily be made with respect to east, west, north and south. These directions could be specified with respect great circles drawn secondary to the prime-vertical. But the Earth's shadow moves along the ecliptic and the Moon is also very nearly moving on the ecliptic at the moment of an eclipse. Thus 'Valana' should give the angle between the ecliptic and the prime-vertical; rather it should be described by two diameters of the Moon's disc, one a secondary to the prime-vertical and one a secondary to the ecliptic. In other words we have to get the angle subtended at the centre of the Moon's disc between those diameters.

This angle between the two diameters mentioned, is, for convenience divided into two parts namely $\hat{KMP}$ and $\hat{PMN}$, where K, P, and N are the poles of the ecliptic, celestial equator and the prime-vertical and M is the centre of the Moon's disc. $\hat{KMP}$ is called Āyana Valana, so called because it depends upon the obliquity of the ecliptic to the Equator (अयनयोः वलनं आयनं वलनम् ie. the deflection due to the deflection of the solsticial points from the equator) whereas the angle $\hat{PMN}$ is called Akṣa Valana ie. deflection of a secondary to the prime-vertical namely NM with respect to a secondary to the celestial equator namely PM which is due to Akṣa or latitude of the place.

We shall first treat the subject on modern lines and then depict Bhāskara's treatment. Let θ, ξ, η stand respectively for the Āyana, Aṣka and total Valanas respectively, where by 'total Valana' we mean $\hat{KMN}$ which is the algebraic sum of $\hat{KMP}$ and $\hat{PMN}$.

From the spherical triangle KMP fig. 78.



$$\frac{\text{Sin } 90 \mp \alpha}{\text{Sin } (90-\beta)} \quad \frac{\sin \theta}{\sin \omega} = \frac{\sin (90-\lambda)}{\sin (90-\delta)}$$

$$\text{Sin } \theta = \frac{\sin \quad \cos}{\cos \delta} \text{ or } \frac{\sin \omega \cos \alpha}{\cos \beta}$$

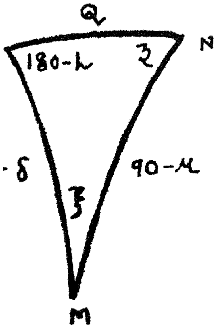


Fig. 79

Similarly from fig. 79, where P=celestial pole, N=North-point, M=centre of the Moon's disc, Q=latitude of the place, h=hour-angle of the Moon, ξ=Āksha Valana and z=Arc of the prime-vertical intercepted between the zenith and the foot of the secondary to the prime-vertical drawn through M, which arc goes by the name Sama-Vritta-Natāmsa or zenith-distance measured along the prime-vertical-μ=distance of M from the prime-vertical measured along the above secondary,

$$\frac{\text{Sin } \xi}{\text{Sin } \varphi} = \frac{\sin (180-h)}{\sin (90-\mu)} = \frac{\sin z}{\sin (90-\delta)}$$

$$\text{Sin } \xi = \frac{\sin \varphi \sin h}{\cos \mu} = \frac{\sin \varphi \sin z}{\cos \delta} \qquad \text{II}$$

In fig. 80, where (M) is the Moon's disc, AB, the diameter of the disc extending along the ecliptic (assuming the Moon's centre almost on the ecliptic, which is the case at the time of an eclipse), K, P, N respectively the pole of the ecliptic, the celestial pole and the north point and θ, ξ the Āyana and Ākṣa Valanas defined above, the eclipse starts at A, the eastern side of AB, called the Krānti-Vritta-Prāchi, AB being perpendicular to EF a diameter of the disc secondary to the ecliptic. An observer with his physical eye construes the diameter CD, which is secondary to the prime-vertical as indicating north and south. Naturally therefore, it is required to specify the

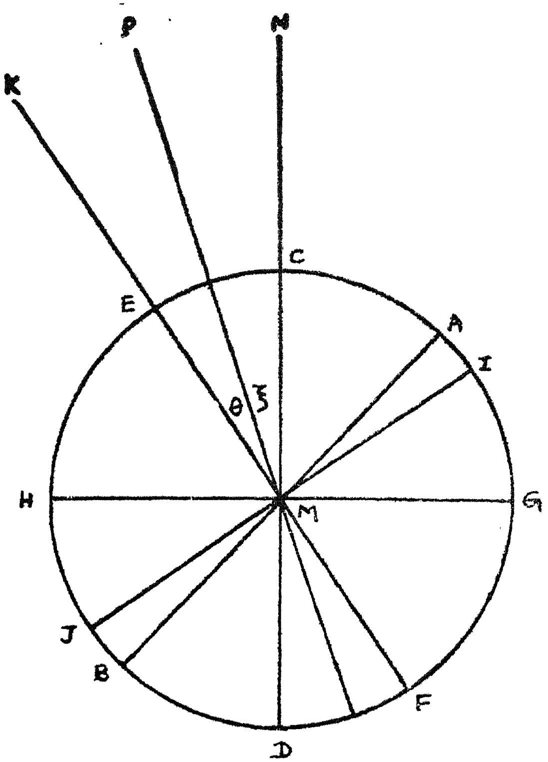


Fig. 80

location of A, the point of first contact, with respect to the diameters GH and CD, which are respectively East-West and North-South. Suppose $\overline{\theta+\xi}=45°=$GMA, then we say that the eclipse begins at the north-point of the disc and so on. For this purpose, the concept of Valana arose. We have said above that the angle $\widehat{KMN}=$GMA is to be got, and that it is the algebraic sum of θ and ξ, meaning thereby that when K comes in between P and N, or below N, which is also likely for places of latitude less

than ω, the obliquity of the ecliptic, $\widehat{KMN}$ will be equal to $\overline{\xi-\theta}$ and $\theta-\xi$ respectively.

The formula given in the present verse is

$$\frac{H}{H \cos \delta}$$

$$\text{or } H \sin \xi = H \sin \left(\frac{90h}{D/2}\right) \times \frac{H \sin \phi}{H \cos \delta}$$

where h and D/2 are measured in nādis, h being the hour angle of the Moon and D/2 half-the duration of the Moon's stay above the horizon.

Evidently the formula is intended as an approximate one, for all practical purposes considered equivalent to formula II given above namely

$$\sin \xi = \frac{\sin \phi \sin z}{\cos \delta} \text{ or } H \sin \xi = \frac{\quad}{H \cos \delta}.$$

Thus in the place of z we are given $\frac{90h}{D/2}$ which means "when z=90°, D/2 is the hour angle measured in nādis, what is z when the hour angle is h?". The answer is

This formula is approximate because h and z are

not strictly in proportion though h increases or decreases along with z.

Nonetheless, the formula serves for practical purposes very approximately and the beauty lies in the concept of Sama-Vritta-Natāmśa, which means measuring hour-angle in terms of the arc of the prime-vertical instead of an arc of the celestial equator. The error, it will be noted will not be much in low latitudes.

So far with respect to the commentary on the present verse. Now we shall see how Bhāskara tackles the problem rigorously in Golādhyāya under the caption Valana Vāsanā' ie. 'concept of Valana'.

We defined above that the angle $\widehat{KMP}$ is Āyana Valana and the angle $\widehat{PMN}$ as the Ākṣa-Valana. These are respectively called Bimbīya-Āyana Valana and Bimbīya-Ākṣa-Valana being subtended at M, the centre of the Bimba ie. the disc of the Moon. If in fig. 78, T be the foot of the celestial latitude of the Moon, then the respective angles $\widehat{KTP}$ and $\widehat{PTN}$ are called the Sthānīya-Āyana-Valana and the Sthānīya-Ākṣa-Valana ie. the angles subtended at the Sthāna or the construed position of the Moon on the ecliptic.

The Āyana-Valana is zero and a minimum when M or T lies at the solstices, and a maximum equal to ω when those points lie at ♈ or ♎. Similarly the Ākṣa-Valana is a minimum equal to zero when M or T lies on the meridian and a maximum equal to ϕ when those points lie at the east or west points. In other words the Āyana-Valana increases from zero to ω as M or T moves along the ecliptic from a solstice to an equinoctial point; and the Ākṣa-Valana increases from zero to the maximum value of ϕ as M or T moves along zE or zω from z the zenith to E or ω along the prime-vertical. Hence Āyana Valana is perceived to be proportional to H sin $(90+\lambda)$ where λ is the longitude of M or T, since when $\lambda = 90$, H sin $(90 + 90) = 0$ and when $\lambda = 0$, H sin $(90+90)=0$ and when $\lambda=0$, H sin $(90+0)=R$, a maximum; similarly the Ākṣa-Valana is perceived to be proportional to H sin z where z is the Sama-Vritta-Natāmśa defined before, since, when $z=0$, M or T is at the zenith and the Ākṣa Valana is zero and when M or T is at the East or West point, $z=90°$ and H sin $z=R$, a maximum.

It is worth-hearing Bhāskara, at this juncture (Ref. verses 30–74. under the caption Valana Vāsanā pages 305–306. Ānandāśrama edition of Golādhyāya Vol. 2. Poona).

"The north and the south with respect to the Equator and Ecliptic (ie. the north-pole and south-pole) are different at the points ♈ and ♎, being at a distance of ω from each other. Hence the Āyana Valanajyā at those points is equal to H sin 24° (ω taken to be equal to 24°). But at the solstices, the north and south will be the same (meaning thereby that the angle subtended by PK at the solstices is zero, or what is the same, the directions to the respective poles (of the Equator and the Ecliptic) at the solstitial points are the same so that the East will be the same for both the circles at those points. Thus there is no Valana at the solstices ie. P♋K = 0 where ♋ = cancer. In between ♈ and ♋, the Valana is found in proportion to H sin $(90+\lambda)$ where λ is the longitude of the point and in inverse proportion to H cos δ, where δ is the declination. Hence H sin $\theta = \dfrac{\text{H sin } (90+\lambda)}{\text{H cos } \delta} \times$ H sin ω, where θ is the Āyana Valana. Similarly at the points of intersection of the Equator and prime-vertical namely E and W, the Unmandala (the Equatorial horizon) decides the north-south direction with respect to the Equator, whereas the horizon decides the same with respect to the prime-vertical. These north-south directions with respect to those two great circles namely the Equator and the prime-vertical differ by the angle between the Unmandala and the horizon which is equal PN = ϕ, the latitude of the place. Hence at the East and West points the Ākṣa Valanajya or the H sine of Ākṣa-Valana is equal to H sin ϕ. But at the zenith, the north-south directions of the Equator and prime-vertical coincide so that there is no Ākṣa Valana at the zenith. Thus H sine of the Valana is proportional to H sin ϕ in between the points on the prime-vertical between the zenith and the East and West points. (Roughly speaking) H sin ξ = H sin ϕ H sin z where ξ = Ākṣa Valana, z :

Natāms'a (defined before) and H sin z may be taken to be roughly equal to $\frac{90h}{D/2}$ (as depicted before).

In the East the Ākṣa Valana is north, for, in fig. 80 $\widehat{GMI}$ which gives the East of the Equator with respect to the the East of the prime vertical, is north; whereas in the West $\widehat{HMJ}$ is south. (The definition of the direction of the Valana is given as a directive to add the two kinds of Valanas if they be of the same direction otherwise to take the difference; in the fig. 80, the Āyana Valana ie. $\widehat{IMA}$ is also north, so that adding $\widehat{GMI} + \widehat{IMA} = \widehat{GMA}$ is the Sphuta Valana or the actual Valana). Hence Sphutā Valana measured by GMA is had by the sum or difference of the two angles $\widehat{GMI}$ and $\widehat{IMA}$ which define respectively the Āyana and the Ākṣa Valanas.

Similarly, at the point of intersection of the Ecliptic and the prime-vertical, the Sphuta Valana is a maximum which is the sum or difference of the Valanas as the case may be. At points removed 90° on either side, from the point of intersection of the Ecliptic and the prime vertical, in as much as the north-south directions with respect to the Ecliptic and the prime-vertical coincide, the Sphuta Valana is zero.

If (as Lallācharya said) *the Valana varies* as the Hversine at those points which are *removed by 90° from the points* of intersection of the Ecliptic and the *prime-vertical, the* Sphuta Valana will not be zero (which is against common sense). Hence the Valanajya varies as Hsine and not as Hversine.

We shall look at the subject from another point of view for the sake of clarity.........Fix a circle on the sphere with the celestial pole as centre and ω as the angular

radius. This circle is called Kadamba-Bhrama-Vritta or the circle in which the polo of the Ecliptic revolves round P (due to diurnal revolution of the Earth). *In that circle Hsine of an angle will be* H sin δ............Or again draw the great circle with the planet's position as the pole, called the horizon of the planet. The arc intercepted on this circle between the Ecliptic and the celestial Equator will be Āyana Valana and that intercepted between the celestial Equator and the horizon is the Ākṣa Valana; and the arc intercepted between the Ecliptic and the horizon is the Sphuta Valana.

Or again draw a circle with K as centre and radius $\omega = 24°$. This circle is called the Jina-Vritta where the word Jina means 24. Let a secondary to the Ecliptic passing through K and K′ the poles of the Ecliptic revolve with KK′ as fixed. When this revolving circle passes through Cancer (Sāyana) it will be passing through P. The angle turned through by this circle from Cancer, will be equal to the angle turned through from P. The Hsine of that angle in the Jina Vritta will be H sin δ of a longitude equal to that angle. This is the Āyana Valana and it arises at the end of Dyujyā, since the north-polar distance of the planet is $(90-\delta)$ whoes Hsine is Dyujya ie. H cos δ. The corresponding Āyana Valana in a circle of radius R is got by multiplying by R and divided by H cos δ

us clarify Bhāskara's mind. (Ref. fig. 81) Let PBD be the Jina Vritta drawn on the sphere with K, the pole of ehe Ecliptic as centre and $\omega = 24°$ as radius. Let a revolving secondary to the Ecliptic coincide initially with K where is Cancer. Let it occupy subsequently the position KM where M is the centre of the Moon's disc taken to be on the Ecliptic as is the case very approximately at the moment of an eclipse. Now the Āyana Valana is the angle KMP. Let MA be the declination of to L such that MK=ML=90°. Hence

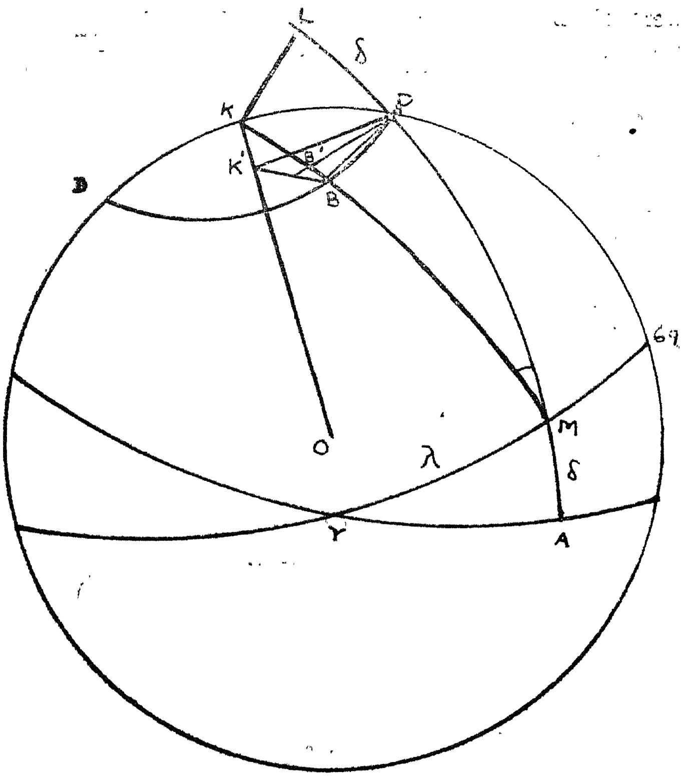


Fig. 81

$=\delta$ since PA$=90°$ and LM$=90°$. The Āyana Valana $\widehat{KMP}$ is measured by the arc ML where ML is an arc of the Grahakṣitija or the horizon of the planet M (ie. the circle with M as centre and $90°$ as radius drawn on the sphere or what is the same the great circle whose pole is M). PB is an arc of the small circle parallel to KL which is an arc of a great circle. Then in the Jina Vritta

sin $\quad = \sin PK \times \sin \widehat{PKB} = \sin \omega \sin (90-\lambda) = \sin \omega \cos \lambda$

$\therefore\ \sin KL = \sin \omega \cos \lambda / \cos PL = \dfrac{\sin}{\cos} \quad = \sin PMK.$

Here $\sin \omega \cos \lambda$ is called Sa-thribbha-graha-ja-kranti or the declination of a point whose longitude is $\overline{90+\lambda}$ where λ is the longitude of M. As we have the formula $\sin \delta = \sin \omega \sin \lambda$, sine of the declination of such a point is equal to $\sin \omega \sin (90+\lambda) = \sin \omega \cos \lambda$. When δ is very small sin PMK may be taken to be $\sin \omega \cos \lambda$ or what is the same Sa-thribbha-graha-ja-krānti as is formulated by Sūrya-siddhānta.

It may be doubted how $\sin PB = \sin PK \sin \overline{90-\lambda}$. (Ref. fig. 82). Let K′ be the centre of the circle PBD,

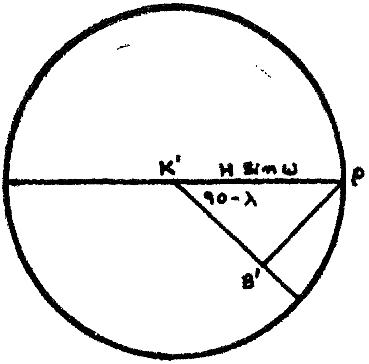


Fig. 82

K′ being in the plane of PBD. K′P and K′B are radii of this circle. Since the arc PB stands for $\overline{90-\lambda}$ $PK'B = 90-\lambda$. Draw the H sine of arc PB, which is PB′. Now $K'P = H \sin \omega$, as PK′ is ⊥ar drawn on OK

$$\therefore \quad PB' = PK' \sin PK'B$$

$$= H \sin \omega \times \sin PK'B = \frac{I}{R}$$

$$\frac{H \sin \omega \; H \cos \lambda}{R}$$

$$H \sin KL = \frac{H \sin \omega \; H \cos \lambda}{R} \times H \cos \delta = \frac{H \sin \omega \; H \cos \lambda}{} \text{ as}$$

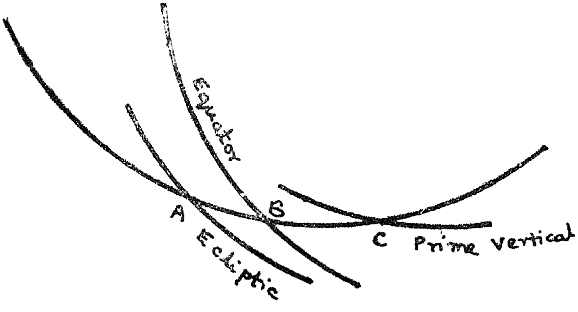


Fig. 83

Note 1. Bhāskara says PB′ (fig. 82) is Krānti-Sinjanī; so it is because $PB' = PK' \sin(90 - \lambda) = \frac{H \sin \omega \; H \cos \lambda}{R}$ = H sine of the declination of a point whose longitude is $\overline{90+\lambda}$ = Satribha-grahaja-krānti as is mentioned by Bhāskara and Sūrya-siddhānta.

Note 2. If M be the centre of the Moon's disc and ABC its horizon defined above, the arc AB intercepted between the Ecliptic and the Equator is Āyana Valana, the arc BC intercepted between the Equator and the prime-vertical is Ākṣa Valana and the arc AC intercepted between the Ecliptic and the prime-vertical is Sphuta-Valana.

Note 3. The analysis of Ākṣa Valana proceeds on similar lines, only we have P and N in the place of K and P.

Note 4. The mistake of Lallācharya alluded to by Bhāskara is as follows. The Āyana Valana, we have seen is zero at the Ayanas ie. the solstices and maximum at ♈ and ♎ ie. the equinoctial points removed by 90° from the Ayanas. Now Hversine = R−H cosine so that when $90 - \lambda = 0$ ie. $\lambda = 90°$, Hversine $(\overline{90-\lambda})$ = R−H cos $(90-\lambda)$ = R−H sin 90° = R−R=0 and when $90-\lambda=90°$ ie.

$\lambda=0$ Hversin $(90-\lambda)$ = R—H cos $(90-\lambda)$ = R—H sin λ = R—H sin 0° = R −0=R. Hence Lallācharya took by mistake that the Āyana Valana varies as Hvers $(90-\lambda)$ instead of H sin $(90-\lambda)$ since both Hversine and H sine of $\overline{90-\lambda}$ are zero at the Ayanas and maximum at ♈ and ♎. The same mistake was committed by Lallācharya in the context of the Moon's phase also as criticised by Bhāskara as we shall see later. In fact, this latter criticism is not so justified as the former, as will be shown in that context.

Note 5. If instead of taking the Āyana Valana to vary as H sin $(90-\lambda)$ we happen to take according to Lallācharya that it varies as Hversine, then in places (Ref. verses 38, 39 Valana Vāsanā, Golādhyāya) removed by 90° from the points of intersection of the Ecliptic and the prime-vertical, where there should be no Sphuta-Valana, we do get that there is some Sphuta Valana there, since the value of Hversine differs from Hsine, though these two functions happen to be zero simultaneously and maximum simultaneously.

Bhāskara continues in verses 66-68 (Ibid) "I shall now depict Ākṣa Valana by means of the hour-angle. Take the sum or difference of S'anku-Agrā and S'anku-tala according as they are of the same direction or not; compute $\overline{R^2-B^2}$ where B is the result; then $\frac{\text{H sin } \phi \text{ H sin } h}{\sqrt{R^2-B^2}}$ is equal to H sin ξ where ξ is the Ākṣa Valana

Comm. We saw before in the Tripras'nādhyāya that A=S+B where A=S'anku-Agrā, S'=S'anku-tala, and B= S'anku-bhuja = H sin μ where μ is represented in fig. 79. Hence $\sqrt{R^2-B^2}$ = H cos μ so that the above formula gives H sin $\xi = \frac{\text{H sin } \phi \times \text{H sin } h}{\text{H cos } \mu}$ which is the same as got by the modern formula in Equation II.

Note. A small circle parallel to the prime-vertical is called Upa-Vṛtta. Also secondaries drawn to the Ecliptic, Equator and the prime-vertical are called Kadamba-Sūtra, Dhruva-Sūtra and Sama-Sūtra. They are also called occasionally as Kadamba-prota-Vṛtta, Dhruva-prota-Vṛtta, and Sama-prota-Vṛtta.

Bhāskara proceeds to find the Āyana Valana in a very ingenious way in verses 69–74. We shall first give it a modern treatment so that we may better appreciate his genius. Let (S) be the Sun's disc. (It does not matter whether we take the Sun or the Moon). EQ is its diameter along the diurnal circle, and CL along the Ecliptic. LM is the difference of the declination of L and S. Let $SL = \Delta\lambda$ and $LM = \Delta\delta$. We have

$\sin\delta = \sin\lambda \sin\omega$;

differentating $\cos\delta\,\Delta\delta = \sin\omega\cos\lambda\Delta\lambda$

$\therefore \quad \Delta\delta = \dfrac{\sin\omega\cos\lambda}{\cos\delta} \times \Delta\lambda$; put $\Delta\lambda = b$, the angular radius of the disc

$$\therefore \quad \Delta\delta = \frac{b \times \sin\omega\cos\lambda}{\cos\delta} = \frac{b\,\mathrm{H}\sin\omega \times \mathrm{H}\cos\lambda}{\mathrm{R} \times \mathrm{H}\cos\delta}.$$

This gives the Valana in the disc of radius b. If that be so, what will it be on the sphere of radius R? The result is $\dfrac{b\,\mathrm{H}\sin\omega \times \mathrm{H}\cos\lambda}{\mathrm{R} \times \mathrm{H}\cos\delta} \times \dfrac{\mathrm{R}}{b} = \dfrac{\mathrm{H}\sin\omega \times \mathrm{H}\cos\lambda}{\mathrm{H}\cos\delta}$ as got before.

Let us hear Bhāskara, "put the disc of the Sun at the point of intersection of the Ecliptic and the diurnal circle. The Valana (LM of fig. 84) at the periphery of the disc is the difference of the declinations of L and S. To get the value of this let us first get the value. SL in terms of λ, the longitude of S. It is $\dfrac{b \times \mathrm{B}}{\ldots}$; so that LM

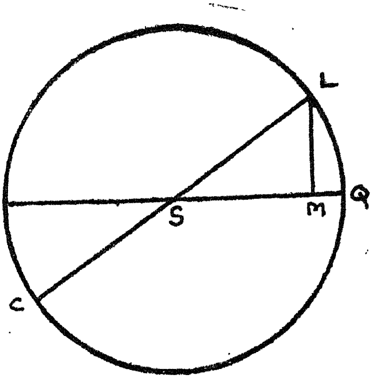


Fig. 84

will be equal to $\frac{b \times B}{225} \times \frac{H \sin \omega}{R}$ where B is the Bhogya-khanda of λ. To obtain the value of the above for a circle of radius R from a circle of radius b, we have to multiply by R/b. So, the result is

$$\frac{b \times B}{225} \times \frac{H \sin \omega}{R} \times \frac{R}{b} = \frac{B\, H \sin \omega}{225}$$

But the value of B is got as follows. 'If for $H \cos \lambda$ equal to R we have the first Bhogyakhanda equal to 225, what shall we have for $H \cos \lambda$?' The result is $\frac{225 \times H \cos \lambda}{R}$ Substituting for B, we have

$$\frac{H \sin \omega}{225} \cdot \frac{225 \times H \cos \lambda}{R} = \frac{H \sin \omega \times H \cos \lambda}{R}$$

Now, on account of declination, the Sun's disc is inclined like an umbrella. So LM of fig. 84 will take a position like L′M as shown in fig. 85 where the triangle MLL′ is similar to SMO, S being the centre of the Sun's disc, O the centre of the sphere. Hence

$$\frac{L'M}{LM} \quad R \quad = \frac{R}{H \cos \delta} \quad \frac{H \sin \omega\ H \cos \lambda}{R}$$

$H \sin \omega \times H \cos \lambda$ as got before'.

Comm. Bhāskara terms SL as the Dorjyāntara' or the variation in H sin λ, which he knows to $\frac{H\ co}{R}$

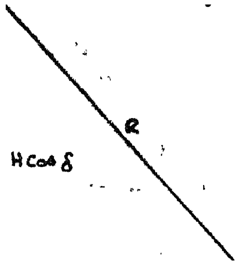


Fig. 85 Fig. 86

But proceeding from first principles, as he always does, he asks us to consider the Bhogyakhanda at λ namely B. If this be for an interval of 225′, what will be it be for 'b'? The result is $\frac{b \times B}{225}$ Then to rectify B, the proportion used is as used above. Bhāskara says many a time that the variation in Hsine is proportional to Hcosine. This concept he might have derived by looking at the Hsine table of 90 Hsines. Hence the argument advanced by him to rectify B is 'If for Hcosine equal to R (at zero-value of the argument) the initial Bhogyakhanda is 225, what will it be for an arbitary H cos λ ? The result is

$\frac{H \cos \lambda}{R} \times 225$. Substituting this for B in the above, we have $\frac{b\, H \cos \lambda}{225} \times 225 = \frac{b\, H \cos \lambda}{R}$. This expression we perceive as no other than $\frac{H}{R}$ as to Δ (H sin λ), for b is to be taken as $\Delta\lambda$. $\frac{b}{R}$ is called by

as Dorjyāntara meaning thereby Δ (H sin λ).

Then the next argument is 'If for H sin λ equal to R we have the declination equal to H sin ω, what shall we have for the above Dorjyāntara. The result is

$\frac{H \sin \omega}{R} \times \frac{H \cos \lambda \times b}{R}$. Since this is in modern terms

$\sin \omega \cos \lambda \times b$ and $b = \Delta\lambda$, we perceive that this expression is Δ (H sin δ) where δ stands for LM of fig. 84, ie. the difference of the declinations of the points S and L of the disc of fig. 84. The next argument advanced by Bhāskara, namely that on account of declination, the disc is slanted and LM gets thereby enlarged into L′M of fig. 84 and adduces proportionality from fig. 86. But this argument seems to be faulty. In fact, the magnitude of LM is got for the diurnal circle of radius H cos δ. To get its value for radius R, the result would be

$$\frac{H \sin \omega \; H \cos \lambda \times b}{R^2} \times \frac{R}{H \cos \delta} = \frac{b.\, H \sin \omega \; H \cos \lambda}{R.\, H \cos \delta}.$$

Then the argument is 'If in the disc of radius b, we have $\Delta\delta$ equal to the above what will it be for radius R? The result would be $\frac{H \sin \omega \; H \cos \lambda}{H \cos}$

Note. Our argument is based on the idea that lines of the small circle namely the diurnal circle get enlarged for a circle of radius R in the porportion R: H cos δ. Bhāskara's concept of enlargement on account of slanting does not seem to be plausible because, on account of declination, the disc may occupy an overhead position when the Equator is itself inclined. Thus slanting does not arise out of declination.

The question might be asked as to how Bhāskara got the right answer by such an argument. He got the answer up his sleeves through the other methods he gave and he adduced this argument to get at that answer.

Second half of verse 21 and first half of verse 22.

$$\frac{\text{H} \cos \lambda \times \text{H} \sin \omega}{\text{H} \cos \delta} = \text{H} \sin \theta \text{ where } \theta \text{ is the Āyana}$$

The direction of this Valana is that of the hemisphere north or south in which the Moon lies.

Comm. This is the formula we have already derived. Regarding the direction of the Āyana Valana, the convention is that it is to be considered north, if the Moon be in the northern hemisphere, otherwise south. The reason is that at the time of a lunar eclipse, the Moon being in opposition, if he be north, the Sun will be south of the Equator, and fhe line BA of fig. 80 representing the Ecliptic which is roughly the join of the Sun and the Moon will be north of the line JI which is parallel to the equator. Thus the direction of the angle IMA gives the direction of the Āyana Valana.

Latter half of verse 22 and verse 23. Sphuta Valana.

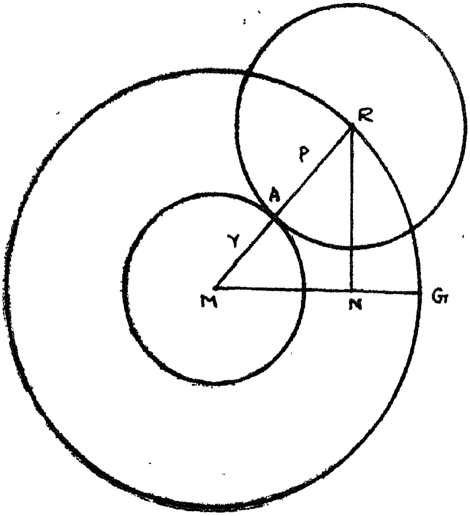
The Hsine of the sum or difference of the two Valanas according as they are of the same or opposite directions, multiplied by the sum of the angular radii of the Moon and Rāhu, and divided by the radius gives the Hsine of the Sphuta Valana. Those who said that the Valana is proportional to the Hversine, do net know spherical geometry properly.

Comm. The direction of the Ākṣa Valana, was defined in verse 20 that it is north if the hour angle is east, otherwise south. The meaning of this convention is that the diameter of the Moon's disc parallel to the Equator when the hour angle is East, is north of the diameter which is parallel to the prime-vertical. Thus combining the two conventions regarding the directions of the Valanas, it is clear that if both the Valanas are north, the line MI is north of MG, and MA is north of MI (fig. 80) so that the

Sphuta Valana is equal to the sum of the angles $\hat{GMI}$ and $\hat{IMA}$. Suppose MA is south of MI either falling within the angle GMI or south of MG, then clearly the Sphutr Valana $\hat{GMA} = \hat{GMI} - \hat{AMI}$ or $\hat{AMI} - \hat{GMI}$ as the case may be, which is obtained as the difference of the two Valanas. Having obtained $\hat{GMA}$ as the Sphuta Valana, H sin GMI multiplied by $(P+r)$ and divided by R gives Hsine of the Sphuta Valana to be represented in a circle of radius equal to $\overline{P+r}$. This latter convention of representing the Sphuta Valana in a circle of radius $\overline{P+r}$ is only a convention. The expression

$$\frac{\text{H sin (Sphuta Valana)} \times \overline{P+r}}{R}$$ gives us RN of fig. 87,

where GMA is Sphuta Valana. In other words, we are to draw fig. 87 to show the point of first contact namely A in relation to MG the line parallel to the prime-vertical.



Verse 24. Conversion of liptas into what are called Angulas.

H cos z of the eclipsed body at the moment of eclipse being divided by the radius and the result being added to $2\frac{1}{2}$ gives the number of liptas per angula. The time elapsed after the rise of the body being divided by the rising hour angle (both being expressed in the same units of time) and the result being added to $2\frac{1}{2}$ also gives the same.

Comm. While a parilekha or a geometrical drawing of the eclipse is attempted at, the problem arises as to how many liptas or minutes of arc giving the measure of the disc are to be taken to be equivalent to one angula. For example, suppose the diameter of the disc is 30′. With what radius shall we draw the disc on a board or paper? In this behalf, a convention based on observations is being mentioned. The discs of the Sun and the Moon are observed to be big at rise and small when they are on the meridian. So, taking the measure of the disc to be 30′ for example, if the eclipse takes place at the rise of the disc, it is laid down to draw the disc with a radius of $15/2\frac{1}{2}$ ie. 6 angulas at the rate of $2\frac{1}{2}$ liptas per angula. {The word angula here mentioned might not be what we take it to be today in our daily parlance as one inch. The gnomon or S'anku was taken in those days to be of a length equal to 12 angulas. Bhāskara mentions in the beginning of Lilavati that 8 Yavas are together equal to one Angula, twentyfour angulas to one hasta, four hastas to one danda and 2000 dandas to one Kros'a, and 4 Kros'as to one Yojana. Also a vamsa is equal to ten hastas. This system discloses that, a Yojana equals 5 modern miles according to Bhāskara's estimate of the diameter of the Earth as compared with its modern estimate. (The method given by Bhāskara as to how the diameter of the Earth could be measured is found to be quite scientific as mentioned by us before).

33 modern inches or angulas are equal to 80 angulas of Bhāskara as per the above.

At this rate the gnomon's modern length would be

Reverting to our subject, we are asked to represent the disc of 30 liptas when the eclipse takes place at noon by $30/3\frac{1}{2}$ angulas counting at the rate of $3\frac{1}{2}$ liptas per angula.

Then the question arises as to what should be the correspondence between the liptas and angulas when the eclipse takes place in between the rise of the Moon (or Sun) and its noon. The directive is that

$$\text{one angula} = 2\tfrac{1}{2}' + \frac{\text{H cos z}}{\text{R}} = 2\tfrac{1}{2} + \cos z \quad \text{I}$$

This means, supposing Z=zenith-distance of the body to be 60°, one angula is to be taken to be equal to $2\frac{1}{2} + \cos 60 = 2\frac{1}{2} + \frac{1}{2} = 3'$ or 3 liptas.

The reason given by Bhāskara reiterating what Sripati said in that behalf, as to why the Moon's or Sun's disc appears to be big at the moment of rising and small on the meridian, is that the disc is immersed in its own rays at noon and rendered small in appearance, whereas, most of the rays are swallowed by the earth or its atmosphere at the moment of rising, making the disc appear large and easily visible.

Note. Bhāskara gives the proof of the above formula I as follows. Since at the time of rise, we are taking $2\frac{1}{2}$ liptas of the measure of the disc to be equal to one angula and while the disc is on the meridian, $3\frac{1}{2}$ liptas are to be taken as one angula, there is an increase of one lipta for an increase of H cos z from zero at the horizon to a value equal to the Radius. So, the argument adduced is 'If for an increase of H cos z equal to the radius, there is an increase

of one lipta in addition to $2\frac{1}{2}$, what should the increase be for an arbitrary H cos z '? The result is

$$\frac{H \cos z}{R} \times 1 = \cos z.$$

Note 2. As finding cos z at the time of an eclipse implies additional arithmetical calculation, and as we have already with us the data of (1) the time elapsed after rise of the celestial body (Moon or Sun) till the moment of the eclipse and (2) half the duration of the day of the body ie. the rising hour-angle converted into time at the rate of 6° per nadi, so a rough formula is given using h in the place of z. The rule of three now used, is 'If for an unnata equal to the dinārdha or rising hour-angle converted into nadis, we have an increase of 1 lipta per angula (over and above $2\frac{1}{2}$ liptas) what shall we have for an arbitrary unnata?' The result added to $2\frac{1}{2}$ liptas, gives the formula one angula $= 2\frac{1}{2} + \frac{\text{Unnata}}{\text{Dinārdha}}$. Dinārdha corresponds to H, the rising hour-angle and Unnata corresponds to (H–h) where h is the hour-angle at the time.

Bhāskara uses the word 'Angula-liptas' meaning thereby the liptas that are to be taken to be equal to one angula while drawing the parilekha of the disc at the time of its eclipse.

Verse 25. Converting Valana etc. into Angulas.

The measures of the Valana (defined above) or the S'ara ie. the celestial latitude of the Moon or the Rāhu Bimba or the Bhuja (defined) are to be converted into Angulas at the rate given by the above formula. While drawing a figure of the solar eclipse, the celestial latitude of the Moon is to be drawn in its own direction whereas in a lunar eclipse, it has to be drawn in the opposite direction.

Comm. The first part is clear. Regarding the second statement, in as much as the centre of the Rāhu Bimba lies

at the foot of the Moon's celestial latitude, the latter has to be drawn in the opposite direction, since in the parilekha, the centre of the Moon's disc is to be at the centre.

Verses 26 to 29. How to depict the eclipse in drawing.

Draw a circle with radius equal to that of the radius of the disc of the eclipsed body and also a circle of radius equal to $r+p$, the sum of the radii of the eclipsed and eclipsing bodies; let directions (east etc.) be marked in the figure. In the outer circle, draw the Valanajya or the Hsine of the Sphutavala with respect to the East point, Valanajyā pertaining to the moment of first contact. In the case of the Moon, the Valanajyā pertaining to the moment of first contact should be marked from the East point and that pertaining to the moment of last contact should be marked from the West point. In the case of the Sun the reverse is to be done. If the Valana is south, it should be marked in the clockwise direction, otherwise anticlockwise.

Having marked the Valanajyā in the form of a Hsine, draw the line joining the centre to the top of the Valanajyā, ie. to the point of intersection of the Hsine with the outer circle. The celestial latitude of the Moon is to be drawn from this top of the Valanajyā in the form of Hsine again. If the latitude pertains to the moment of first contact, it should be drawn from the top of the Valanajyā pertaining to that moment, and if it pertains to the moment of last contact, it should be laid off from the top of the Valanajyā pertaining to the moment of last contact. The celestial latitude pertaining to the middle of the eclipse should be drawn from the centre along the line of Valanasūtra or the line joining the centre to the top of the Valanajyā. Taking the extremities of these latitudes, circles are to be drawn with the radius of the eclipsing body to depict the eclipse at the respective moments.

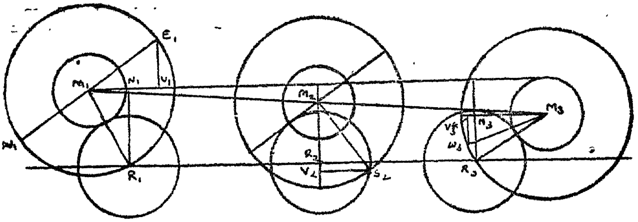


Fig. 88

Comm. Let M_1, M_2, M_3 be the positions of the centre of the Moon's disc at the moment of first contact, at the middle of the eclipse and the moment of last contact respectively. Draw a circle with radius $M_1 R_1 = r+p$ which is called the Manaikyārdha Vritta. Let $M_1 E_1$ represent the Eastern direction known as the Samamandalaprāchī. Draw $E_1 V_1$ equal to Hsine of the Valana, so that $M_1 V_1$ is the Krānti-Vritta-prāchī ie. the point of intersection of the Ecliptic with the Eastern horizon. Draw $N_1 R_1$ perpendicular to $M_1 V_1$ in the form of Hsine, which is the latitude (Vikṣepa) of the Moon at the moment of first contact. With R_1 as centre draw the Grāhaka-Vritta with p as centre; this circle represents the Rāhu-Bimba.

Similarly let M_3 be the centre of the Moon's disc at the moment of last contact. Draw a circle with M_3 as centre and $r+p$ as the radius which is the Manaikyārdha-Vritta. Let $M_3 W_3$ be the direction to the West, the Samamandalapratīchī. From W_3 draw $W_3 V_3$ the Hsine of the Valana, so that $M_3 V_3$ is now the Krāntimandala-pratīchī ie. the point of intersection of ecliptic with the western horizon. Let, the Vikṣepa $N_3 R_3$ be drawn as a Hsine of the Mānaikyārdha Vritta. With R_3 as centre and radius p, draw the Rāhu-Bimba.

Let M_2 be the centre of the Moon's disc at the middle of the eclipse. Let $M_2 S_2$ represent the South with respect to the prime-vertical. From S_2, draw the Hsine of the Valana $S_2 V_2$ so that $M_2 V_2$ is the Krānti-Vritta-Dakshinā ie. the South with respect to the Ecliptic. Now the

Vikṣepa or the celestial latitude of the Moon $M_2 R_2$ is to be drawn along this Valanasūtra M_2 V. With R_2 as centre and radius p drawn the Rāhu-Bimba.

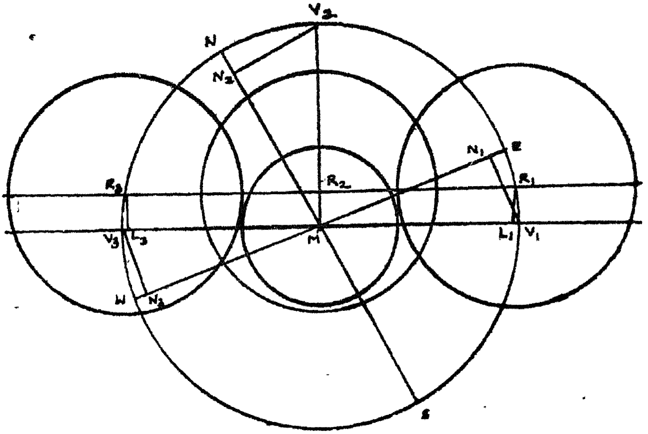


Fig. 89 Depiction of Fig. 88, keeping the Moon fixed

The slight flaw in this figure is that $ML_1 L_3$ implied as the path of the Moon is taken to be parallel to the ecliptic $R_1 R_2 R_3$ the path of the eclipsing body the Grāhakamārga, in as much as latitudes are drawn perpendicular to $ML_1 L_3$.

This figure depicts a total eclipse of the Moon. If M coincides with R_2 at the middle moment of the eclipse, then the eclipse is called central.

The duration of a central eclipse will be on the average the time that the Moon's disc takes to cross the diameter of the Rāhu-Bimba with its relative velocity. Hence the mean duration of a central eclipse is

$$\frac{\text{Average diameter of Rāhu}+\text{Average diameter of the Moon}}{\text{Relative velocity of the Moon with respect to the shadow}}$$

$$= \frac{(81+64)\times 24}{790'\text{-}35''-59'\text{-}8''}\text{ hrs} = \frac{145}{732}\times 24 = \frac{290}{61} =$$

4 hrs–45 minutes approximately.

Verse 30 *and first half of* 31. Geometrical depiction of the eclipse at the beginning and end of totality and also of the magnitude of the eclipse.

The Bhuja is to be laid from the centre of the Moon along its Valanasūtra or the line indicating the direction of the ecliptic; the latitude is to be drawn from the end of the Bhuja and perpendicular to the Bhuja. The hypotenuse is to be drawn from the centre of the Moon. Taking the point of intersection of the latitude (Kōti) and the hypotenuse, as centre and radius p equal to that of the eclipsing body, if circles be drawn, from these circles could be known the points where totality begins and ends as well as the magnitude of the eclipse at any given moment. Or these could be found in another way as follows.

Comm. The method given above for depicting the phases of an eclipse geometrically, could be applied for any moment during the course of the eclipse and depends upon before-hand computed Bhuja and Kōti. Refer to fig. 90. Let M be the centre of the Moon's disc. Mark Eω the East-west line drawn through M. Compute the Valana for the required moment, either for the moment when totality begins or for that when totality ends or for

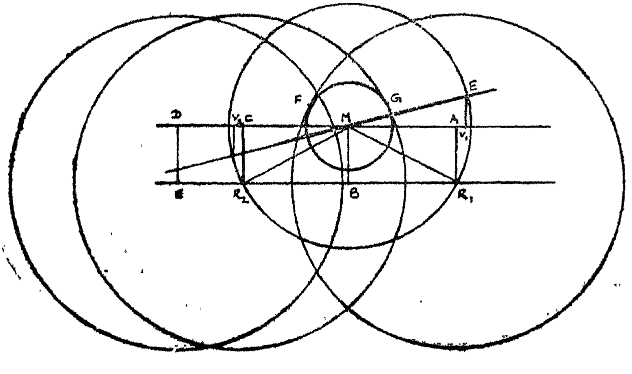


fig. 90

any arbitrary moment whatsoever. With this Valana primarily laid in the Manaikyārdha Vritta, decide the Krānti Vrittaprachī or the East-West direction of the eclipse. Thus in the figure V_1 V_2 is this direction. Then lay off the computed Bhuja along this V_1 V_2 from M, say MA for the Sammīlana moment or the moment when totality begins or MC for the Unmilana moment or the moment when totality ends or MD for an arbitrary moment. Draw AR_1 or CR_2 or DE equal to the latitude at the particular moment, perpendicular to the Valanasūtra. In the figure drawn the Valanasūtra is shown to be the same. This does not mean it will be the same throughout. It will be changing because the position of the Ecliptic changes from moment to moment. So Bhāskara uses the word ie. 'the respective Valanasūtra'. Also the latitudes will be differing from moment to moment as well as the Bhujas both of which are to be computed for any moment along with the Valana. (The method of computing the Bhuja was given in verse 15). Computing the respective Karṇas or the hypotenuses from the formula $K = \sqrt{Bhuja^2 + K\bar{o}ti^2}$, (Kōti is here the latitude) with centre M and radius equal to the Karṇas, if arcs be drawn to cut the latitudes, the points of intersection would be no other than R_1, R_2 or E. Join MR_1, MR_2 and ME. With centres R_1 and R_2 and radii equal to $p-r$, (where p is the radius of the Rāhu-Bimba, and r the radius of the Moon's disc) if circles be drawn, they just touch the Moon's disc at F and G which are the points where totality begins and ends respectively. With centre E and radius P, if a circle be drawn, that will show what amount of the disc is shadowed as well as the measure of the magnitude of the eclipse (defined in verse 11).

Note. In the above commentary and figure we have depicted MD as the Iṣta-Bhuja or the Bhuja at a given moment, taking a moment prior to the Unmīlanakāla, for showing the magnitude of the eclipse.

If a moment in between the Sammīlana and Unmīlana were taken, the then Bhuja and Koṭi could be no doubt computed, but the question of magnitude of the eclipse does not arise as the entire disc has been plunged in the shadow.

Second half of verse 31 *verse* 32 *and first half of verse* 33. Alternative method of depicting the eclipse geometrically.

Joining the upper end of the latitude of the middle moment of the eclipse to those of the first and last contacts, we have what are called the Pragrahamārga and Mokṣamārga ie. the path of the centre of the eclipsing body from the first contact to the middle moment of the eclipse and that from the middle moment to the last contact. The lengths of these paths could be computed and they could be drawn before hand. Then with the centre of the Moon as centre and radius equal to $p-r$, if a circle be drawn, it cuts the paths described above each in one point. With those points as centre and radii equal to p, if circles be drawn, they will touch the Moon's disc each in one point which are respectively the points of Sammīlana and Unmīlana.

Comm. In as much as the latitude of the Moon differs from moment to moment, the Pragrahamārga and the Mokṣamārga are separated to achieve a little more accuracy than could be got by joining the upper extremities of the initial and final latitudes. The remaining statement is evident, for, at the moments of Sammīlana and Unmīlana, the distance between the centres of the eclipsing body and the eclipsed will be $p-r$, so that the points of intersection of the Pragrahamārga and Mokṣamārga with the circle whose centre is the centre of the eclipsed body and radius $p-r$ will give the centre of the eclipsing body.

Latter half of verse 33. To know the magnitude of the eclipse at any given moment during the course of the eclipse.

Let the product of the time elapsed from the moment of first contact and the length of the path of the eclipsing body traced from the moment of the first contact to the middle of the eclipse divided by the time between the moment of first contact and the middle of the eclipse, be x. Similarly let the product of the time before the end of last contact and the path of the eclipsing body traced between the middle moment of the eclipse and the moment of last contact divided by the time between the middle moment and the moment of last contact be y. Lay off x and y units of length from the first and last points of the path of the eclipsing body along the path respectively. Then we get the points of the centre of the eclipsing body at the required moments. With these points as centre and radius p, if circles be drawn, they represent the eclipsing body. The length of the diameter of the eclipsed body shaded, gives the magnitude of the eclipse called grāsa.

Comm. Here rule of three is applied namely "If during time T_1 or T_2 a path equal to l_1 or l_2 in length is traced what length will be traced in times t_1 or t_2?", where T_1 and T_2 are the times called Sparśa-Sthiti-Khanda and Mokṣa-Sthiti-Khanda respectively, l_1 and l_2 are the times elapsed from the moment of first contact or before the moment of last contact and t_1, t_2 are the times from the beginning of the eclipse and before the end of the eclipse respectively. Then x and y give the points where the centre of the eclipsing body lies.

Verse 35. Given the magnitude of the eclipse at any time to obtain the time elapsed after the first contact.

The time taken by the centre of the eclipsing body to move through the segment of the path of the eclipsing

body which lies between the position of the eclipsing body at the moment of first contact and the point of intersection with the path of the eclipsing body of the circle drawn with the centre of the eclipsed body as centre and radius equal to the difference of $p+r-g$ where g is the magnitude of the eclipse (grāsa) at the moment, or similarly the time taken by the centre of the eclipsing body to move through a similar and equal segment of the path of the eclipsing body on the other side, gives the time elapsed after the moment of first contact or the time before the moment of last contact.

Comm. This is the converse of the above problem. The method is clear being based on rule of three as above. Both the problems could be algebraically expressed as follows. Let T, t, l, g, and k, stand respectively for the Sthiti-Khanda ie. the time between the moment of first contact to the middle of the eclipse or the time between the middle moment to the moment of last contact ; (2) the time elapsed after the moment of first contact or the time before the moment of last contact, as the case may be ; (3) the length of the Pragrahamārga or Mokṣamārga ; (4) the grāsa which is defined as $p+r-k$; (5) the Karṇa whose expression is $\sqrt{B^2+\beta^2}$, B being the Bhuja defined and β the latitude of the Moon.

Then the following working is stipulated (a) If in time T, a path of length l is traced, what will be traced in t? The result is $\frac{lt}{T}$ (b) Then $B = l - \frac{lt}{T}$ (c) $B^2 + \beta^2 = K^2$ (d) $p+r-k=g$. Thus combining all the steps

$$l^2 (T-t)^2 + \beta^2 T^2 = T^2 (p+r-g)^2 \quad I$$

given t, this equation gives g and given g it gives t.

Again the following relation holds good between T and l, $l^2 = (p+r)^2 - \beta^2$ II and

$\frac{l}{m_1-s_1}$ = T with the nomenclature already employed which means $T = \frac{\sqrt{(p+r)^2-\beta^2}}{m_1-s_1}$ III

In the above working, the fundamental elements are p, r, β, m_1 and s_1 with which the other elements could be worked out. Replacing the other elements from equation I, we have

$$\{(p+r)^2-\beta^2\} \left\{\frac{\sqrt{(p+r)^2-\beta^2}}{m_1-s_1} - t\right\}^2 + \beta^2 \left(\frac{\overline{p+r^2}-\beta^2}{(m_1-s_1)^2}\right) = \frac{(p+r)^2-\beta^2}{(m_1-s_1)^2} (p+r-g)^2$$

ie. $\{(p+r)^2-\beta^2\} [\sqrt{(p+r)^2-\beta^2}-t(m_1-s_1)]^2 + \beta^2(\overline{p+r^2}-\beta^2) = (\overline{p+r^2}-\beta^2)(p+r-g)^2$

ie. $\{\sqrt{\overline{p+r^2}-\beta^2}-t(m_1-s_1)\}^2 + \beta^2 = (p+r-g)^2$ IV

Putting $t=0$ in this equation, we have $(p+r)^2 = (p+r-g)^2$ ie. $g=0$ which means at the moment of first contact, the grāsa is zero. Again putting

t = T ie. $t = \frac{\sqrt{(p+r)^2-\beta^2}}{m_1-s_1}$ ie. $t(m_1-s_1) = \sqrt{\overline{p+r^2}-\beta^2}$

we have

$\beta^2 = \overline{p+r-g^2}$ ie. $g = p+r-\beta$ which gives the grāha at the middle of the eclipse which was defined as the Sthagita. In equation IV which we may take as a fundamental equation, the two unknowns are t and g one of which being given the other could be got.

Verse 36. The colour of the eclipe.

When less than half the disc of the Moon is eclipsed, the colour will be what is called Dhumra ie. of the colour

of smoke; when the disc is half eclipsed, the colour is black; when more than half is eclipsed, the colour would be a blend of black and red and when the entire disc is eclipsed, the colour will be what is called pis'anga or reddish-brown.

Comm. Clear.

Verse 37. When declare the occurrence of an eclipse.

When even one-sixteenth of the diameter of the Moon's disc is shadowed, the eclipse will not be visible in as much as the shadowed portion is covered by the illuminating rays of the disc. In the case of the Sun, when even one-twelfth of the diameter is shadowed, the eclipse will not be visible for the same reason. Hence we shall not declare the occurence of an eclipse upto the shadowing of the discs to the extents stated above.

Verses 38 and 39. Examples which disclose the invalidity of construing Valana in terms of Hversine instead of Hsine.

When the Sun is in the zenith, the Ecliptic being vertical, the Valana is clearly seen to be the Agra of ($\odot+90$) where $\odot$ is the longitude of the Sun. If you could show that the Valana will be the same on the basis of Hversine–formula, then I would accept that what Lallāchārya postulated in his work Sişya-Dhī-Vṛddhida is correct.

Again, in a place of latitude $90-\omega$, ω being the obliquity of the Ecliptic (ω is taken to be 24°), when the Sun being situated in Meşa, Vṛşabha, Mīna or Kumbha, the Moon contacts him from the south at the moment of a solar eclipse, in as much as the Ecliptic coincides with the horizon. In this circumstance, how could the Valana be equal to R, as made out by the Hversine–formula.

Comm. Lallāchārya gave the Valana in terms of the following verses " स्पष्टाद्रिकालजनितोत्क्रमशिञ्जिनीर्वर्णेन
कृ क्रमेण; ग्राह्यात् सराशित्रितयात् भुजक्रमज्या..." Verses 23, 25 Chandragrahaṇādhikāra, wherein he formulated the Valana in terms of Hversine in the place of Hsine. The reason for his slip, we have already explained. Now Bhāskara gives two glaring examples to substantiate his formula and to show up the flaw in Lallācharya's formulation.

In the first example, where the Ecliptic takes the form of a Vertical, the Sun being in the zenith, the Spaṣṭa Valana which is the angle between the Ecliptic and the prime-vertical is the same as the arc between the East point and the intersection of the Ecliptic with the horizon known as Lagna. Since the Sun is then in the zenith, the longitude of the Lagna is $(90+\odot)$ so that the said arc is the Agra of the point whose longitude is $\overline{90+\odot}$ as stated. Hence Spaṣṭa Valanajyā = sin A = sin δ/cos ϕ where A is the agrā (using Napier's rule from triangle PNL where L is the Lagna N the north-point and P the celestial pole). In the Hindu form, this is given by $\text{H sin V} = \text{H sin A} = \frac{\text{R H sin } \delta}{\text{H cos } \phi}$ where δ is the declination of a point of the Ecliptic whose longitude is $(90+\odot)$. But Lallāchārya's formula gives the Valanajyā as Hvers δ, δ being the declination of a point of the Ecliptic whose longitude is $(90+\lambda)$, λ being the longitude of the Eclipsed body ignoring the latitude. In other words, in the case of the lunar Eclipse when the Moon is in the zenith his Valanajyā = Hvers δ (δ having the above value) the Ākṣa Valanajyā here being zero. Since $\frac{\text{R H sin } \delta}{\text{H cos } \phi} \nless$ Hvers δ, the mistake committed by Lallāchārya is evident even supposing H cos ϕ = R when we ignore the latitude ie. take ϕ to be zero.

In the second example cited by Bhāskara the Ecliptic coincides with the horizon, the pole of the Ecliptic being in the zenith. Then in a Solar Eclipse the Moon eclipses the Sun from the south showing that H sin V = R. That H sin V = R is also evident from the fact that the Ecliptic makes 90° with the prime-vertical, having coincided with the horizon. But here according to Lallāchārya's formula, H sin ξ = Hvers 90° = R and Āyana Valanajyā is Hvers δ, where δ is the declination of a point whose longitude is 90° more than $\odot$. If $\odot = 30°$, 60° H sin θ =

$$\frac{R}{2}\sin\omega/R \text{ or } \frac{\sqrt{3}}{2}\frac{R\sin\omega}{R} \text{ ie. } \frac{\sin\omega}{2} \text{ or } \sqrt{3}/2\sin\omega.$$

Evidently the sum of the two Valanas Āyana and Ākṣa cannot be 90° as is also vouchasafed from geometry. So, here also, the flaw is evident.

Note 1. Srīpatyāchārya also followed Lallāchārya vide verses 18, 19, 20 Chandragrahaṇādhyāya, Siddhānta Sēkhara. It will be noted that the commentator of Siddhānta Sēkhara, while reiterating Bhāskara's stand as the correct one, himself commits a mistake in saying In fact

$$\sin\xi = \frac{\sin\phi\sin h}{\cos\mu} = \frac{\sin\phi\sin z}{\cos\delta} \text{ as proved by us before.}$$

The commentator cited above overlooked that sin ξ could be also equal to $\frac{\sin\phi\sin h}{\cos\mu}$, wherein natakāla also is implied.

Note 2. It will be noted that even Pṛthūdakāchārya, while commenting on Brahmasphuta Siddhānta, ignored Brahmagupta and followed Lallācharya blindly.

Note 3. The formula given by Lallāchārya and followed by Pṛthūdaka as well as by Srīpati is very rough besides containing the flaw cited, in as much as both μ and δ are taken to be zero, which are not so.

Verse 1. In as much as the observer situated on the surface of the Earth and as such elevated by the radius of the Earth from the centre there of, perceives not the Sun and the Moon having the same longitude at the moment of conjunction, to be in the same line of sight, heyt being depressed unequally having different orbits, so I proceed to elucidate what are called Lambana and Nati ie. parallax in longitude and latitude, on which account they are not in the same line of sight.

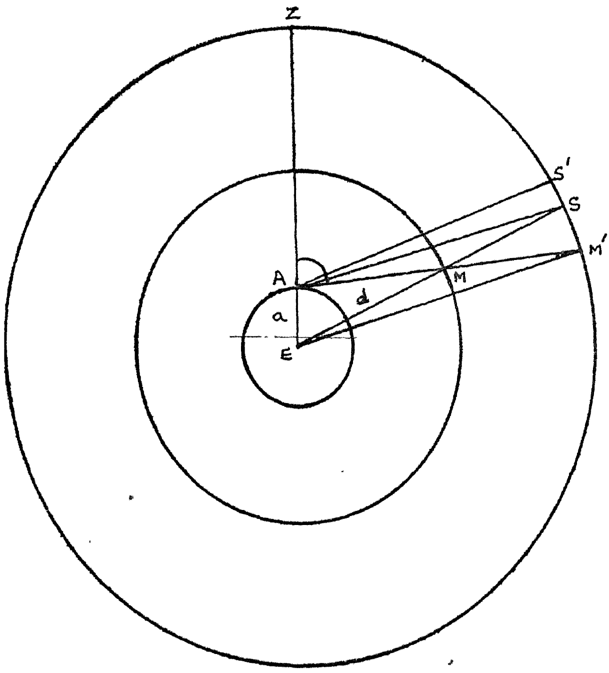


Fig. 91

Comm. (Refer fig. 91) Let E be the centre of the Earth, M and S the centres of the discs of the Moon and

the Sun. Let A be the position of an observer on the surface of the Earth, elevated by the radius EA from E. Let M and S be in the same line of sight as seen from E. But as seen from A, AS and AM are respectively the lines of sight to the Sun and the Moon. Evidently these lines of sight differ the Moon being depressed more than the Sun. If a line AS′ be drawn which is parallel to the central line of sight namely EMS, we find that the Sun is depressed by the angle S′AS whereas the Moon is depressed by the angle S′AM′. These angles differ because the orbits of the Sun and Moon differ.

Here the angle S′AS will be very very small, its magnitude being in truth just about 8″ only. But the angle S′AM′ will be sufficiently large since the Moon is very near the Earth compared with the Sun. Taking ES and AS to be almost parallel due to the largeness of the Sun's distance, the angle $S\hat{A}M$ will be almost equal to $A\hat{M}S$ so that we could consider that the Moon is depressed from AS the line of sight to the Sun by the angle SAM′ = $A\hat{M}E$. This angle AME is called the geocentric parallax of the Moon and the angle $A\hat{S}E$ that of the Sun $M'\hat{A}S$ = angle of depression of the Moon over and above that of the Sun = M′AS′ − M′AS = $E\hat{M}A - E\hat{S}A$ = geocentric parallax of the Moon minus geocentric parallax of the Sun.

Verse 2. The presence or absence as well as the positiveness and negativeness of the parallax in longitude.

Compute the Lagna at the moment of conjunction of the Sun and the Moon. There will be no parallax in longitude when the Sun is situated at the point called Vitribha or the point whose longitude is ≡ L − 90°, L being the longitude of the Lagna ie. the ascendant which is the point of intersection of the Ecliptic with the

horizon. If the Sun's longitude falls short of the longitude of the Vitribha or exceeds it, there will be parallax in longitude which will be positive in the former case and negative in the latter.

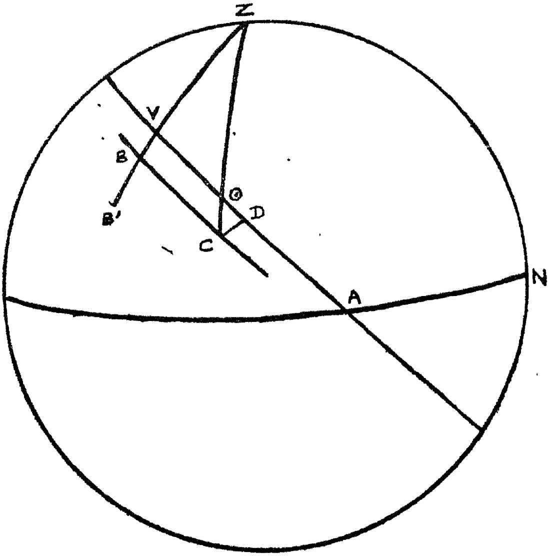


Fig. 92

Comm. (Ref. fig. 92) Let SN be the horizon, Z the zenith and VA the Ecliptic. A is the ascendant or Lagna. Let V be the point called Vitribha which is 90° behind A. Strictly speaking V is called Vitribhalagna or lagna from which three Rāśis or 90° are subtracted (Bha=Rāśi. त्रिभिर्विरहितम् वित्रिभम्; वित्रिभम् च तत् लग्नम् च वित्रिभलग्नम् ie. a point whose longitude is got by subtracting three Rāśis from that of the Lagna). Let ZV be the vertical of V so that $\widehat{ZVA} = 90°$. It will be seen that $AV = 90°$ as follows. Let A′ be the point where the Ecliptic intersects the horizon on the west. One will construe that the Ecliptic is bisected by the meridian; but it is not so. Spherical triangles AVZ and A′VZ being right-angled at

V are congruent because AZ = A'Z and ZV is common $\therefore$ AV = VA'. But AV + VA' = 180° because the Ecliptic and the horizon being two great circles, they bisect each other. Hence AV = 90°. Then a celestial body situated at V will be depressed along ZV the vertical, say, to a point B. Let ⊙ be any arbitrary position of the Sun; then ⊙ will be depressed along the vertical Z⊙, say, to a point C. Draw CD perpendicular on the Ecliptic. Then ⊙D is the component of the parallax ⊙C along the Ecliptic where as DC is its component perpendicular to the Ecliptic. Thus ⊙D is the parallax in longitude and DC is the parallax in latitude. The word 'Lambana' means etymologically लम्बते अनेनेति लम्बनम् ie. that amount by which the celestial body is depressed (along the Ecliptic). In Hindu Astronomy the word Lambana is applied to parallax in longitude alone whereas the word Nati is applied to parallax in latitude. Hence to translate Lambana as parallax alone is not correct. The word Drik-lambana is applied to mean parallax along the vertical, and the word Sphutalambana is occasionally used to connote parallax in longitude.

As Bhāskara rapidly comments on the verses in this Gaṇitādhyāya, he having dealt with the subject of parallax elaborately under the caption, Grahaṇa Vāsanā, in the Golādhyāya, to catch up his thought, we have to treat the subject first from the modern view point and then elucidate what he has said in the Golādhyāya, much matter of which is reiterated by him under the commentary here in the Gaṇitādhyāya.

(Ref. Fig. 91) From the $\triangle$EAM,

$\overline{\quad a} = \quad d \quad\quad d$

the Earth and d the distance of the celestial body (here the Moon)

$\therefore$ Sin $\widehat{EMA} = \frac{a}{d} \sin \widehat{ZAM} = \widehat{EMA}$ expressed in radian measure since $\widehat{EMA}$ is very small

$\therefore$ $\widehat{EMA}$ (expressed in radian measure) $= \frac{a}{d} \sin z$ I where z is the apparent zenith-distance of the Moon ie. zenith-distance as seen by the observer (in contradistinction to the geocentric zenith-distance of the Moon namely $\widehat{ZEM}$). In particular, when $z = 90°$, $\widehat{EMA} = \frac{a}{d}$ which is the maximum parallax known as the horizontal parallax ie. the parallax when the Moon is situated on the horizon of the observer. Also the parallax is zero when M is situated at Z as is seen from formula I and as is rightly remarked by Bhāskara in the words '

In fig. 91, $\widehat{EMA}$ is the angle by which the line of sight of the observer namely AM is depressed from the geocentric line of sight EM. Since the plane of the paper represents a vertical through the Sun and the Moon, the depression of either the Sun or the Moon or the excess of the depression of the Moon over the Sun are all in the vertical plane. This depression is called Drik-lambana because it is a lambana or depression in the Drik-maṇḍala or vertical.

This Drik-lambana varies as sin z as is seen from formula I where a, and d may be taken to be constants. (Both a and d vary slightly a varying slightly from place to place on the Earth, the Earth being an oblate spheroid, and d varying from position to position of the Moon).

The maximum horizontal parallax is given by $\frac{a}{d}$ in radian measure which is equal to, according to Bhāskara's

estimate $\frac{1581}{2} \times \frac{1}{51566} \times 3438 = 53'$ approximately. Its modern value is about $57'$ so that the Hindu estimate is not far from truth. Here 1581 and 51566 are the values of a and d in Yojanas according to Bhāskara.

The Hindu astronomers do not, however, proceed exactly as we have done in the para above to obtain the maximum horizontal parallax. Their treatment is a little different and is as follows. Whereas according to Modern astronomy $\widehat{EM'A}$ (fig. 93) is viewed as the horizontal

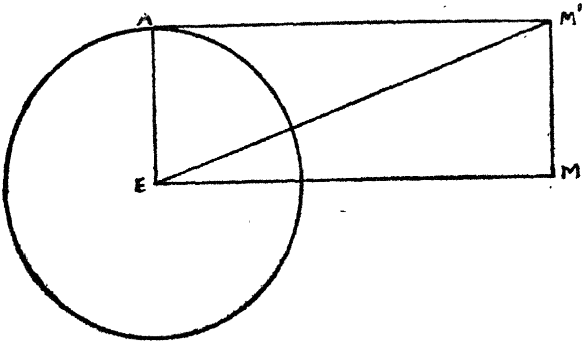


Fig. 93

parallax, in Hindu Astronomy $\widehat{MEM'}$ or the angular measure of the Moon's path equal to the radius of the Earth is taken to be the horizontal parallax. Both, of course, mean the same as is seen from the figure.

In as much as the Hindu astronomers knew very well what they term कलीकरण or converting linear distances into angular measure, converting a linear magnitude equal to the radius of the Earth namely MM' at the lunar orbit into angular measure, they got

$\frac{MM'}{EM} \times 3438 \quad \frac{1581}{2} \times \frac{}{51566}$ radians $= 52'\text{-}42''$ as the maximum horizontal parallax.

Having got this estimate, they reckoned this angular measure in time as the time taken by the Moon to traverse the distance MM′ equal to the radius of the Earth as follows. The Moon traverses 790′-35″, which is exactly 15 times 52′-42″. Hence they said that the maximum horizontal parallax is $\frac{1}{15}$th of the Moon's daily motion in arc and expressing it in terms of time, that the maximum horizontal parallax is $\frac{1}{15}$ of a day or $\frac{1}{15}$th of 60 nādis or 4 nādis.

That this horizontal parallax is 4 nādis as a maximum, would have been also verified at the time of a solar eclipse when the Sun was situated on the horizon at the time of conjunction, by the fact that the eclipse occurred four nādis in advance of the moment of geocentric conjunction (which could be calculated very accurately by the Hindu astronomers, as could be seen by the very correct estimate of a lunation in Hindu Astronomy. In fact, the length of a lunation must have been estimated correctly by noting the time-interval between two solar eclipses or lunar and by dividing that time by the integral number of lunations elapsed in between the two eclipses).

The question then arises as to how the Hindu Astronomers could know the distance of the Moon. From the estimate of the horitontal parallax by actual observation, and from the geometry of fig. 93, a correct estimate of the distance of the Moon must have been arrived at.

Having thus known that the Moon traverses a distance equal to the radius of the Earth in 4 nādis, his daily linear motion was estimated to be 15 times the radius of the Earth ie. $\frac{15 \times 1581\frac{1}{6}}{2} = 11858\frac{3}{4}$ Yojanas.

The daily motion of the Moon having thus been estimated almost correctly, an act of inexpedience on the part of the Hindu astronomers was that they should have

presumed that all the other planets including the Sun would be traversing the same linear distance during the course of a day. This led to a wrong estimate of the Sun's distance as well as his spherical radius. Also, they supposed wrongly that the parallax of the Sun also would be equal to $\frac{1}{15}$th of his daily arcual motion.

Their estimate of the spherical diameter of the Moon was, however, very near the truth, for, they argued, that if 790′-35″ angular motion per day corresponded to $11858\frac{3}{4}$ Yojanas in linear measure, to what linear measure did the angular diameter of the Moon namely 32′-0″-9‴ correspond? The answer was

$$\frac{11858\frac{3}{4} \times}{790'\text{-}35''}$$

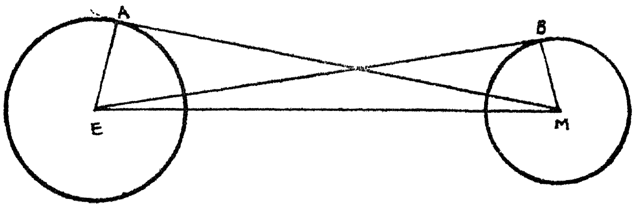


Fig. 94

It may be here pointed out that there is a relation between the angular radii of two celestial bodies as seen from each other and their mutual horizontal parallaxas (Fig. 94). Let E and M bet he centres of the Earth and the Moon respectively. $\widehat{EMA}$ = horizontal parallax of the Moon = angular radius of the Earth as seen from the Moon and $\widehat{BEM}$ = Angular radius of the Moon = Horizontal parallax of the Earth as seen from the Moon. Thus, we see that the Earth will be seen from the Moon, as a Moon with an angular radius equal to 57′. In other words our Earth will be a Moon to our Moon, having nearly 16 the area of our Moon's disc.

The periphery of the Moon's orbit was arrived at as follows. "If 790′-35″ of the Moon's angular motion corresponds to $11858\frac{3}{4}$ Yojanas to what periphery must $360 \times 60'$ correspond?" The answer is

$$\frac{11858\frac{3}{4} \times 360 \times 60}{790'\text{-}35''} = 324000$$ Yojanas. Reverting to the subject of parallax on hand, the Drik-lambana or the parallax along the vertical has the formula $\frac{4 \text{ H sin z}}{R}$ I

nādis in Hindu Astronomy where 4 nādis is the maximum parallax obtained when H sin z = R.

From fig. 92, $C\odot^2 = CD^2 + D\odot^2$ ie.

Drik-lambana² = Nati² + Sphuṭalambana² II

$$CD = \odot C \sin C\hat{\odot}D = \frac{4 \text{ H sin z}}{D}$$

$$\sin C\odot D = 4 \sin z \sin V\hat{\odot}Z$$

$$= 4 \sin \odot Z \sin V\hat{\odot}Z = 4 \sin ZV$$

Thus, the parallax in latitude at any point of the Ecliptic is that at the Vitribha which is conveyed by Bhāskara in the words "कक्ष्योरन्तरं यत स्यात वित्रिभे सर्वं

$$\text{Also } \odot D = C\odot \cos C\hat{\odot}D = 4 \sin \odot Z \cos Z\odot V \quad \frac{[\sin V\odot}{R^2} \qquad \text{IV}$$

= Maximum parallax × Vitribha-S'anku × H sine of the arc V⊙.

In the above working we proceeded in a modern way. It is worth-hearing Bhāskara as to how these results were arrived at elegantly and ingeneously from first principles.

In fig. 91, EMS is called Garbha-Sūtra whereas AS is called Dṛṣṭi-Sūtra.

ie. In as much as the Moon is depressed from the Dṛk-Sūtra, so this phenomenon goes by the name **Lambana.** It will be noted that in Hindu Astronomy geocentric parallax will not be treated separately for the Moon and the Sun but dealt with simultaneously as it is called for, in the context of a solar eclipse. They were interested in knowing the relative depression of the Moon with respect to the Sun rather than knowing the separate magnitudes with respect to the Moon and Sun, for which they had no application.

ie. In as much as the Garbha-Sūtra and Dṛk-Sūtra are identical in the direction EAZ (fig. 91) there is no parallax at the zenith.

Now consider the plane through ZV of fig. 92. Suppose the EMS of fig. 91 is in the direction EV. Then both the Sun and the Moon may be considered to have the same Vitribha at that moment of conjunction. Both the Sun and the Moon being then depressed along ZV, to V′ and V″ respectively the Ecliptic will then be a circle parallel to VA (fig. 92) through V′ and the orbit of the Moon will be another circle parallel to VA through V″, V″ being below V′. If we neglect, for a moment, the depression of the Sun, and consider VD to be the Ecliptic on which the Sun is situated undeflected, and BC to be the deflected orbit of the Moon relative to the Sun, then VB is the Nati of the Moon, which will be the same distance between VD and BC, ie. the orbits of the Sun and the Moon.

This fact was proved by us analytically in the modern way showing that CD = BV.

This Nati it is that influences the latitude of the Moon, which may cause apparent conjunction when there is no geo-

centric conjunction and which does not show an apparent conjunction when there is a geocentric conjunction. In other words parallax in latitude plays a very important part in solar eclipses. Also ⊙D being the parallax in longitude, the moment of apparent conjunction might be preceded or followed by a geocentric conjunction according as the Sun lies along VA or AV. Thus having determined the exact moment of apparent conjunction using the magnitude of ⊙D, then we have to rectify the latitude using the magnitude of VB. If that rectified latitude of β falls short of $R+r$ where R is the angular radius of the Sun and r that of the Moon, then there will be a solar eclipse.

It will be noted that when the Sun coincides with V at the moment of conjunction, there is no parallax in longitude ⊙D being zero (Fig. 92) in that position. Also there will be no parallax in latitude when the Ecliptic assumes the position of a vertical circle passing through the zenith, the Drik-lambana then being entirely along the Ecliptic. In this case the Vitribhalagna V will coincide with Z and $\frac{4 \text{ H} \sin \text{V}\odot}{\text{R}}$ which is termed the Madhyamalambana is now entirely along the Ecliptic and as such it is the Sphutalambana in this case. We have said above that when V coincides with Z, the Madhyamalambana is zero at Z, and that the maximum is equal to 4 nādis on the horizon. In between Z and the horizon it has the formula $\frac{4 \text{ H} \sin \text{V}\odot}{\text{R}}$. Noting further that in this case when the H cosine of the zenith-distance of V ie. the S'anku of V is R, the entire lambana is along the Ecliptic, the nāti being zero, and the Madhyamalambana is itself the Sphuta-lambana, and again when V does not coincide with Z, H cos ZV is no longer R but has assumed Kōti-Rūpa ie. the form of a H cosine, as well as the Sphuta-lambana also, which assumes Kōti-Rūpa ie. of the form ⊙D of Fig. 92, where ⊙C is the Madhyamalambana, ⊙D is

Kōti = Sphuta-lambana, and DC = Bhuja = Nāti, it is argued that the Sphuta-lambana is proportional to H cos ZV, assuming a maximum value when V coincides with Z (⊙ not being at Z).

Verses 3 and 4. Parallax in longitude based on two proportions.

Compute the H cosine of ZV, by calculating the rising time of AV, the Kujyā, Dyujyā and Antyā pertaining to V, as was formulated in the Tripraśnādhikāra, then H sin V⊙, multiplied by 4 and divided by R, and again multiplied by H cos ZV and divided by R again gives the parallax in longitude.

Comm. As per the above formula, parallax in longitude equal to ⊙D of Fig. 92 is equal to

$\frac{4 \text{ H sin V}\odot \times \text{H cos ZV}}{R^2}$. This is evidently derived out of two proportions that the parallax in longitude is proportional to H sin V⊙ as well as H cos ZV. This we have already derived through modern methods as formula IV.

Under verse 2. The two proportions are (1) V coinciding with Z, if by H sin V⊙ equal to R, we have 4 nādis as the maximum lambana on the horizon, what shall we have by an arbitrary H sin V⊙? The result is

$\frac{4 \text{ H sin V}\odot}{R}$ and (2) V not coinciding with Z, if by H cos ZV equal to R we have $\frac{4 \text{ H sin V}\odot}{R}$ as the Madhyamalambana, what shall we have for an arbitrary H cos ZV? The result is

$\frac{4 \text{ H sin V}\odot}{R} \times \frac{\text{H cos ZV}}{R}$ as formulated.

First half of verse 5. Alternate method of rectifying lambana. The Madhyamalambana multiplied by 12 and

divided by the Chāyākarṇa of the Vitribha will also give the Sphuta-lambana.

Comm. From Tripraśnādhikāra, we have

$\frac{12}{K} = \frac{H\cos Z}{R}$ so that in the formula cited above instead of

$\frac{H\cos ZV}{R}$ we are asked to use 12/K.

Latter half of verse 5 and first half of verse 6.

$$\text{Dṛk-nati}^2 = H\cos^2 ZV - H\cos^2 Z\odot$$

$$= H\sin^2 Z\odot - H\sin^2 ZV \text{ (fig. 92)}$$

$$\frac{4\ \text{Dṛk-nati}}{R} = \text{Sphutalambana.}$$

Comm. We shall first prove this on modern lines.

$$\text{Cos } Z\odot = \cos ZV \cos Z\odot$$

$$\therefore \quad \cos^2 ZV - \cos^2 Z\odot = \cos^2 ZV\,(1-\cos^2 V\odot)$$

$$= \cos^2 ZV \sin^2 V\odot = \frac{H\cos^2 ZV\ H\sin^2 V\odot}{R^4}.$$

Also $\cos^2 ZV - \cos^2 Z\odot$

$$= \sin^2 Z\odot - \sin^2 ZV =$$

$$\frac{H\cos^2 ZV - H\cos^2 Z\odot}{R^2} = \frac{H\sin^2 Z\odot - H\sin^2 ZV}{R^2}$$

$$\therefore \quad H\cos^2 ZV - H\cos^2 Z\odot = H\sin^2 Z\odot - H\sin^2 ZV$$

$$= \frac{H\cos^2 ZV\ H\sin^2 V\odot}{}$$

$\therefore$ Dṛk-nati defined above $= \sqrt{H\cos^2 ZV - H\cos^2 Z\odot}$

$$\overline{\sin^2 Z\odot - H\sin^2 ZV} = \frac{H\cos ZV\ H\sin V\odot}{R}$$

$$\therefore \quad \frac{4\ \text{Dṛk-nati}}{R} = \frac{4\ H\cos ZV\ H\sin V\odot}{R^2}$$

= Sphutalambana.

Bhāskara's proof proceeds in two stages from first principles. In the first place when V coincides with Z, Lambana is seen to be equal to 4 nādis on the horizon and zero at Z ie. it is zero when H sin Z⊙ = 0 and 4 nādis, a maximum when H sin Z⊙ = R. So, it is meet that Lambana should be taken to be proportional to H sin Z⊙ ie. proportional to natajyā. In this context the lambana termed as Madhyamalambana is entirely along the ecliptic. It is taken to be in the form of Karṇa, because in the position of ⊙C also it is in the form of a Karṇa.

Then let the Ecliptic be deflected from the zenith (deflected = क्षिप्त). Vitribhalagna then being deflected from the position of Z, occupies the position of V (fig. 92). So ZV is called Dṛk-kshepa since the Ecliptic which was in the form of a Dṛk-mandala is deflected from that position. Also the circle ZV is called Dṛk-kshepa-mandala because V is deflected along that circle. Now consider the △ whose sides are H sin ZV, H cos ZV and R. H cos ZV equal to R and as such in the form of a Karṇa corresponds to the Madhyamalambana which is also in the form of a Karṇa; when this Vitribha-S'anku assumed the form H cos ZV, ie. rendered a Kōti from its form of a Karṇa, R, the Sphutalambana is also rendered a Kōti in the form of ⊙D so that

$$\frac{\text{Madhyamalambana}}{R} = \frac{\text{Sphutalambana}}{H \cos ZV}$$

$$\text{Sphutalambana} = \frac{H \cos ZV}{R} \text{Madhyamalambana}$$

$$ZV \times 4 \text{ H sin } Z\odot$$

Then Bhāskara says that we could look at this, from another angle in the words "यदेव स्फुटलम्बनस्य कोटिरूप

In fig. 92, H sin Z⊙ is in the form of a Karṇa; H sin ZV is in the form of the corresponding Bhuja. This triangle formed by these two as sides may be taken to be similar to the triangle ⊙DC, both being called parallax △s. This plane triangle ⊙DC is like the plane triangle which has for its sides H sin λ, H sin δ, where λ and δ are the longitude and declination of a point of the Ecliptic.

In the Triprasnādhikara, we had occasion to deal with this triangle and there we had R

$$\frac{\sqrt{\text{sin}^2 \lambda - \text{H sin}^2}}{\text{H cos } \delta} \quad - \quad \text{right}$$

sion of the point. Similarly R $\dfrac{\sqrt{\text{H sin}^2 \text{Z}\odot - \text{H sin}^2 \text{ZV}}}{\text{H cos ZV}}$

= H sin V⊙. In other words Dṛk-nati is H sin V⊙ projected into a circle of radius H cos ZV from a circle of radius R. We have the proportion

$$\therefore \frac{\text{C}\odot}{\text{H sin Z}\odot} = \frac{\text{CD}}{\odot \text{ sin ZV}} = \frac{\text{D}\odot}{\sqrt{\text{H sin}^2 \text{Z}\odot - \text{H sin}^2 \text{ZV}}}$$

$$= \frac{\text{D}\odot = \text{Sphutalambana}}{\sqrt{\text{H sin}^2 \text{Z}\odot - \text{H sin}^2 \text{ZV}}}$$

The quantity under the radical in the denominator is called Dṛk-nati for the following reasons.

When V coincides with Z, V⊙ is the Dṛk-mandalanata, so that when V is deflected also, we continue to view the Dṛk-nati placed along V⊙. Since Madhyamalambana ⊙C is in the form of a Karṇa in the △⊙DC, we perceive it to be in the form of a Karṇa even when V coincides with Z. This Madhyamalambana being equal to Sphutalambana when V coincides with Z, Sphutalambana is also in the form of a Karṇa then. Now in the position ⊙DC, Sphutalambana has assumed the position of a Kōti ie. the Sphutalambana which, in the form of a Karṇa, being placed along Dṛk-mandala natāmsa, is now

rendered a Kōti and is placed along V ⊙, So the quantity $\sqrt{H \sin^2 Z\odot - H \sin^2 ZV}$ which is the Kōti of the △ formed by H sin Z⊙ and H sin ZV, corresponds to the Kōti of Sphuta-lambana ⊙D. So we call $\sqrt{\sin^2 Z\odot - H \sin^2 ZV}$ as Dṛk-nati, in as much as the Sphutalambana being placed along V⊙ in the form of a Karṇa when V⊙ is Dṛk-mandala-nata, continues to be placed along V⊙ in the deflected position also and becomes a Kōti corresponding to the Kōti of the triangle formed by H sin Z⊙ and H sin ZV ie. corresponding to the quantity $\sqrt{H \sin^2 Z\odot - H \sin^2 ZV}$. The Sphuta-lambana should be construed as being associated with Dṛk-mandala-nata which term is now abbreviated to the term Dṛk-nati.

At A of fig. 92, the Dṛk-nati $= \sqrt{R^2 - H \sin^2 ZV}$ $= H \cos ZV = H \cos ZV =$ Vitribha-lagna-S'anku. Hence the proportion proceeds in accordance with this Dṛk-nati.

Verse 6 (*latter half*) *and first half of verse* 7. Alternative method of obtaining parallax in longitude.

$$\left(\frac{H \cos ZV}{/4}\right)^2 - \left(\frac{H \cos Z\odot}{R/4}\right)^2$$

or

$$\left(\frac{\quad}{R/4}\right) - \left(\frac{\overline{V}}{R/4}\right)^2$$

gives the parallax in longitude expressed in nādis.

Comm. These formulae just constitude another mode of expressing the parallax in longitude and the equivalence of the formulae with the formula $\frac{4}{R}$ Dṛk-nati is evident.

Latter half of verse 7. Use of the parallax in longitude.

The time of the ending moment of New Moon ie. the of geocentric conjunction is to be rectified by this

parallax in longitude to get the moment of apparent conjunction by the method of successive approximation.

Comm. In as much as the moment of apparent conjunction for an observer situated somewhere on the surface of the Earth precedes or follows the moment of geocentric conjunction being preponed or belated by the parallax in longitude, we have got to take this parallax in longitude into account and compute the moment of apparent conjunction. This computation has to proceed according to the method of successive approximation since the hourly motions of the Sun and the Moon vary as well as the parallax in longitude. When the Sun is in advance of V, the Sphutalambana advances the Moon more than the Sun so that the moment of apparent conjunction is past. Hence the correction is negative and vice versa.

Verses 8 and 9. Computation of the parallax in longitude without an appeal to the method of successive approximations.

Let Para $= \frac{13}{32}$ H cos ZV ; $\{\text{Para} \sim \text{H sin} \odot \text{L}\}^2 +$ H $\cos^2 \odot$ L $= \text{K}^2$ H $\sin^{-1}\left\{\frac{\text{H cos} \odot \text{L} \times \text{Para}}{\text{K}}\right\} =$ parallax in longitude.

Ref. fig. 95, $E_1 E_2$ is taken to be what is termed Para equal to $\frac{4}{R}$ H cos ZV, H cos ZV being the Vitribha-S'anku.

Since $\frac{4}{R}$ could be written as $\frac{\text{H sin } 24}{R}$, since 4 ghatis $=$ $\frac{4 \times 360}{60} = 24°$, 60 ghatis being equivalent to 360°,

$\frac{4}{R}$ H cos ZV $= \frac{\text{H sin } 24}{R} \times$ H cos ZV. Imagining for a moment H cos ZV has come in the place of H sin λ

pertaining to the formula $H \sin \delta = \frac{H \sin \lambda \times H \sin 24}{R}$

$\frac{4}{R}$ H cos ZV = Para = the H sine of the declination of that point whose longitude is equal to Vitribha-S'anku. In other words Para is termed as the Vitribha-S'anku-Rūpa-Krānti-Vṛttiya-Bhujajyājanita-Krāntijyā.

Now take $E_1 E_2$ = Para defined above. Draw circles of equal radii with E_1 and E_2 as centres. Call (E_1) and (E_2) as the Chandra-Kakshāmandala and Ravi-Kaksha Mandala. Para by its formulation as $\frac{4}{R}$ H cos ZV, is equal to the maximum parallax in longitude for a given H cos ZV ie. for a given position of V with respect to Z. This being so, the parallax in longitude for an arbitrary position of ⊙ with respect to V will be

$\frac{\text{Para} \times H \sin (\odot - v)}{R}$ according to the previous lation thereof. This form of the formula by its similarity with the formula $\frac{a}{R}$ H sin m, pertaining to the eccentric-circle-theory, suggested to Bhāskara that the parallax in longitude could be derived from the theory of the eccentrics or Prati-Vṛtta-Bhangī. In fig. 95, it will be noted that E_1, E_2 are not the centre of the Earth and the position of the observer on the surface of the Earth but such points as $E_1 E_2$ is made equal to $\frac{4}{R}$ H cos ZV or $\frac{a}{d}$ H cos ZV of the modern figure $\frac{4}{R}$ being equal to $\frac{a}{d}$, so that $E_1 E_2$ is of a variable magnitude varying with H cos ZV.

Comm. When H cos ZV = R ie. when V coincides with Z, we have the maximum parallax. What then will

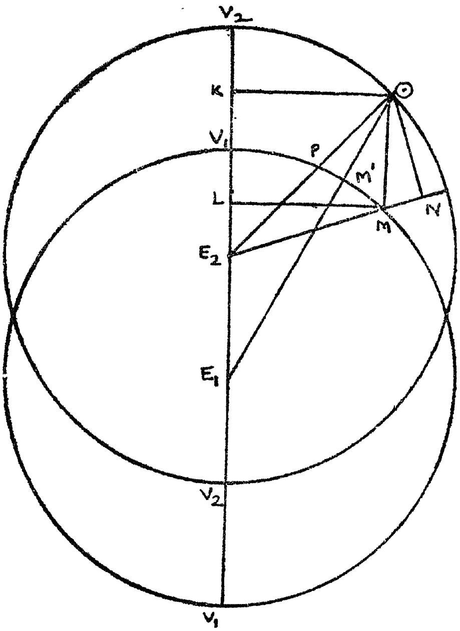


Fig. 95

be had for an arbitrary H cos ZV? The result is $\frac{\text{H cos ZV}}{3438} \times$ H sin 24 since 4 nādis correspond to 24°, 60 nādis corresponding to 360°. Hence the result is $\frac{\text{H cos ZV} \times 1397}{3438}$. Converting $\frac{3438}{1397}$ into a continued fraction we have $2 + \frac{1}{2+} \frac{1}{5+} \frac{1}{1+} \frac{1}{9}$ ......of which the convergents are $\frac{2}{1}$, $\frac{5}{2}$, $\frac{27}{11}$, $\frac{32}{13}$ and $\frac{32}{13}$ is a very good convergent preceding a large quotient namely 9. So the result may be written as $\frac{13 \text{ H cos ZV}}{32}$ which is symbolized as

Now parallax in longitude $= \frac{\text{Para} \times \text{H sin} (\odot - v)}{\text{R}}$.

When H sin $(\odot - v) =$ R, the parallax will be equal to

Para. This formula by its similarity with the formula pertaining to the eccentric theory led Bhāskara to use the method of eccentric circles to obtain the parallax. It is indeed ingenious on his part to have conceived the applicability of that method.

Further it is rather curious that 4 nādis of the maximum parallax should correspond to 24°. This also led Bhāskara to conceive similarity between the formulae $H \sin \delta = \frac{H \sin \lambda \times H \sin 24°}{R}$ (the formula used to obtain the declination δ given the longitude λ of a point of the Ecliptic) and the formula $\frac{H \cos ZV \times H \sin 24}{R}$ = Para. So from an arbitrary H sin λ equal to Vitribha-S'anku, Para is derivable as H sin δ. In other words Para is called Vitribha-S'anku·Rūpa-Krānti·Vṛttiya Bhujajyā-Janita-Krāntijyā.

Now the doubt arises, namely that when the formula longitudinal parallax $= \frac{\text{Para} \times H \sin(\odot - v)}{R}$ resembles the formula $\frac{a}{R}$ H sin m which pertains to the Equation of centre, why does Bhāskara suggest that the parallax is derivable without the application of the method of successive approximations, by appealing to the method S'īghra-phala. The doubt is here two fold (1) where is the necessity for the method of successive approximation to obtain the parallax, though it be called for, to obtain the moment of conjunction? (2) why does Bhāskara appeal to S'īghrakarma and not Mandaphala, when the formula suggests the latter, by the presence of R and there is no K at all?

The answer is as follows. In the first place, even in the modern formula for parallax namely a/d sin z, Z is

the zenith-distance pertaining to the observer and not the geocentric zenith-distance, which are respectively called pṛṣṭhīya and garbhīya natāṃsas. Also the parallax is the angle between the geocentric direction of the Moon and that of the observer. (Vide fig. 91 where parallax = $\widehat{EMA}$).

In deriving this parallax, we are using the apparent zenith-distance of the Moon and not the geocentric zenith-distance of the Moon. In fig. 92 the position of ⊙ corresponds to the geocentric position, whereas D corresponds to the position of the observer on the surface of the Earth. So, as we use the apparent zenith-distance as argument to obtain parallax along the vertical, so we have to use, VD as the argument to derive the parallax in longitude and not V⊙. So, the method of successive approximations is called for as ⊙D is first computed from the argument V⊙ and VD is to be made the argument thereafter. This means that V⊙ may be construed as Madhyakēndra and VD as Sphutakēndra. Now applying this idea to fig. 95, $V_2 E_2 \odot$ may be construed as Sphutakēndra whereas $V_2 E_1 \odot$ may be construed as Madhyakēndra.

From the similarity of the triangles ⊙NM, and E_2LM, $\frac{\odot N}{LM} = \frac{\odot M}{E_2M}$ $\therefore$ $\odot N = \frac{\odot M}{E_2M} \times LM =$

$$\frac{\text{Para}}{\text{K}} \times \odot K = \frac{\text{Para}}{\text{K}} \times \text{H sin } \widehat{KE_2\odot}$$

where E_2M is termed the Karṇa and $\widehat{KE_2\odot}$, the Sphutakēndra is made the argument. Thus parallax in longitude which was originally formulated as $\frac{\text{Para} \times \text{H sin } \overline{\odot - v}}{\text{R}}$ (in which case, the method of successive approximation was called for), is now formulated as $\frac{\text{Para}}{\text{K}} \times \text{H sin } (KE_2\odot)$

where that method of successive approximation is circumvented and where by the presence of K in the place of R, analogy is with the eccentric method of formulation of Sīghraphala and not that of Mandaphala. Also

$$K^2 = E_2M^2 = E_2L^2 + ML^2 = (E_2K - LK)^2 + \odot K^2 = (E_2K - M\odot)^2 + \odot K^2 = (H \cos \widehat{KE_2\odot} - \text{Para})^2 + H \sin^2 \widehat{KE_2\odot}.$$

But if L be the lagna of the moment $L\odot = 90 - V\odot$ so that $H \cos \widehat{KE_2\odot} = H \sin \odot L$ and $H \sin \widehat{KE_2\odot} = H \cos \odot L$ $\therefore K^2 = (H \sin \odot L - \text{Para})^2 + H \cos^2 \odot L$ as formulated in the verse.

Fig. 95 is in the plane of the Ecliptic. The parallax in the vertical circle is projected on to the plane of the Ecliptic by taking $\frac{4}{R}$ H cos ZV as the Para, and deriving the parallax in longitude from this Para.

Now, the doubt arises as to why the Sīghroccha is not taken to coincide with the Vitribha but is taken as removed 180° therefrom.

Verse 10. H sin ZV (of fig. 92) is called the Dṛk-kshepa of the Sun, which is considered to be north in case the northern declination of the Vitribha is greater than ϕ the latitude, otherwise south.

Comm. Let in fig. 96, AV be the Ecliptic whereof A is the ascendant or Lagna and V the Vitribhalagna. Let EQR be the celestial Equator. Let δ be the declination of the Vitribhalagna. Then if $\delta > \phi$. then ZV, the arc of the the Dṛk-kṣepa, (H sin ZV being defined as the Dṛk-kṣepa) as well as H sin ZV are considered to be north. Thus in fig. 96, it is north whereas in fig. 97 it is south. (In fig. 97, *r* is shown outside the celestial sphere, signi-

fying that r is in the western hemisphere and is brought into view for clarity).

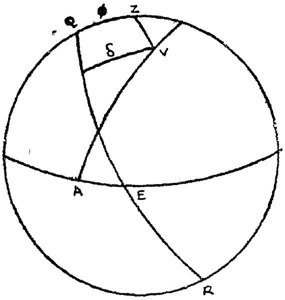


Fig. 96

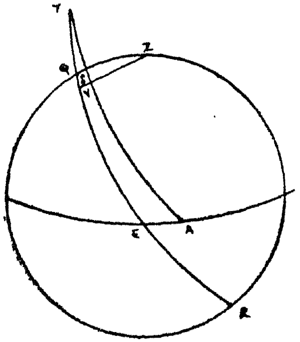


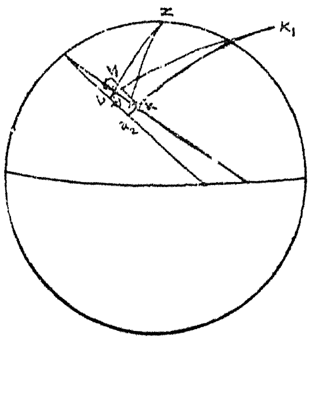
Fig. 97

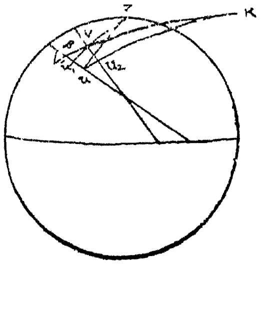
Verse 11 *and first half of verse* 12. Then the sum of ZV and the latitude of V assuming V to be the Moon, or the difference of the above two, as the case may be, according as both of them are north or of opposite directions, gives the arc whose Hsine is the Dṛk-kṣepa of the Moon, The Dṛk-kṣepas of the Sun and the Moon multiplied respectively by $\frac{1}{15}$th of their daily motions and divided by the radius R (equal to 3438′) are the parallaxes of the Sun and the Moon in latitude. The sum or difference of these parallaxes according as they are of opposite or the same direction, is the true parallax in latitude in the context of a solar eclipse.

Comm. The true parallax in latitude sought above is the relative parallax of the Sun and the Moon in latitude. Suppose in fig. 92, VB is the parallax in latitude pertaining to the Sun and VB′ that pertaining to the Moon; then BB′ is the relative parallax, the difference being taken in this case because both are of the same direction.

Parallax in latitude namely CD in fig. 92, we saw equal to VB which is equal to $\frac{4H}{R}$ sin ZV, In other words

the parallax in latitude either of the Sun or the Moon is equal to $\frac{4}{R}$ H sin of the zenith-distance of the respective Vitribhalagna wherever the Sun or the Moon be situated in their orbfts namely the Ecliptic or the Vimandala. Let H sin ZV be the Dṛk-kṣepa of the Sun, V being the Vitribhalagna pertaining to the Sun and let H sin Zv be the Dṛk-kṣepa of the Moon where v is the Vitribhalagna of the Moon. (Ref. figures 98 and 99) Let K′ be the pole of the Vimandala and vv_1 the latitude of v. Since v and V are in the proximo, the latitude of v may be taken to be very nearly equal to the latitude of V so that ZV $\pm$ latitude of V is very nearly equal to Zv. In fig. 98, ZV—latitude of V is very nearly equal to Zv because both ZV and latitude of V are of the same direction. In fig. 99 ZV+latitude of V is very namely equal to Zv because both arc of opposite direction. Thus Zv = ZV $\pm$ latitude of V approximately and H sin ZV and H sin Zv are the Dṛk-kṣepas of the Sun and the Moon respectively. Having got these Dṛk-kṣepas $\frac{4}{R} \times$ Dṛk-kṣepa gives the nati in each case ie. the parallax in latitude and the sum or difference of these natis as mentioned in the beginning of the commentary of this verse gives the relative parallax of the Moon with respect to the Sun which is called the





true parallax in latitude. This true parallax in latitude increases or decreases the latitude of the Moon at the moment of conjunction as is going to be mentioned in the latter half of verse 14.

In deriving the parallax in latitude from the respective Dṛk-kṣepas, instead of using the formula $\frac{4}{R}$ Hsine (Dṛk-kṣepa) which is an expression in time, it is sought to express the same in arc because the latitude of the Moon is expressed in arc and we have to take the sum or difference of the latitude and the parallax in latitude to obtain the apparent latitude of the Moon at the moment of conjunction. In the case of the parallax in longitude we sought to express the same in time because the moment of apparent conjunction was sought therefrom.

Latter half of verse 12 *and first half of verse* 13. An approximate method of obtaining the relative parallax in latitude of the Moon with respect to the Sun.

The Hsine of the zenith-distance of the nonagesimal pertaining to the Moon or what is called the Moon's Dṛk-kṣepa multiplied by 2 and divided by 141, gives the relative parallax in latitude of the Moon with respect to the Sun; or working with the smaller table of Hsines (where R is taken to be 120) the Moon's Dṛk-kṣepa being multiplied by 2 and divided by 5 and the result being increased by $\frac{1}{60}$th of itself gives approximately the relative parallax in latitude.

Comm. Herein, the Vitribha or the nonagesimal of the Sun is taken to concide with that of the Moon. In other words the Dṛk-kṣepas (the Hsines of the zenith-distances of the nonagesimals, of both the Sun and the Moon are taken to be identical. Then using the following proportion "If by a Dṛk-kṣepa equal to R, the relative parallax in latitude is equal to $\frac{1}{15}$th of the difference of the

daily motions namely 48′–46″, what will it be for an arbitrary Dṛk-kṣepa ?" We have

continued fraction, this will be equal to

$\frac{1}{70+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{10+}\,\frac{1}{3}$ of which a very approximate convergent is $\frac{2}{141}$ as taken by Bhāskara.

If the radius be taken to be 120, the coefficient of D will be $\frac{48\frac{3}{4}}{120} = \frac{195}{480} = \frac{13}{32} = \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{6} = \frac{2}{5}$ very approximately.

Latter half of verse 13 *and first half of verse* **14.** An easy method to compute the parallax in longitude and latitude.

Taking the Dṛk-ṣepa of the Moon as well as the Sun to be the Hsine of the meridian zenith-distance of the Vitribha and the H cosine of its meridian zenith-distance as the Vitribha-S'anku, the parallaxes in latitude and longitude could be got from them respectively.

Comm. Parallax in longitude is computed from the Vitribha-S'anku, whereas parallax in latitude is computed from the Dṛk-kṣepa or the Hsine of the zenith-distance of the Vitribha. Thus for both the purpose the Vitribha's position is important, whose zenith-distance and altitude give respectively the parallax in latitude and longitude. Since in practice it is a little cumbrous to obtain the Vitribha's altitude and zenith-distance, an approximate procedure is suggested. Obtaining the declination or the Sphuta-krānti of the Moon taking him to coincide with the Vitribha by the method described in verse 3 of the Graha-cchāyādhikāra, and using the formula $z+\delta=\phi$, the meridian zenith-distance of the Vitribha can be got. This may be assumed to be the Dṛk-ṣepa approximately. The

complement of the meridian-zenith-distance may be assumed to be the Vitribha-S'anku approximately. Then the parallaxes in latitude and longitude could be computed respectively from the two as described before.

The following figure gives a particular nomenclature that was in the mind of Kamalākara, the author of Siddhāntatattvavivēka.

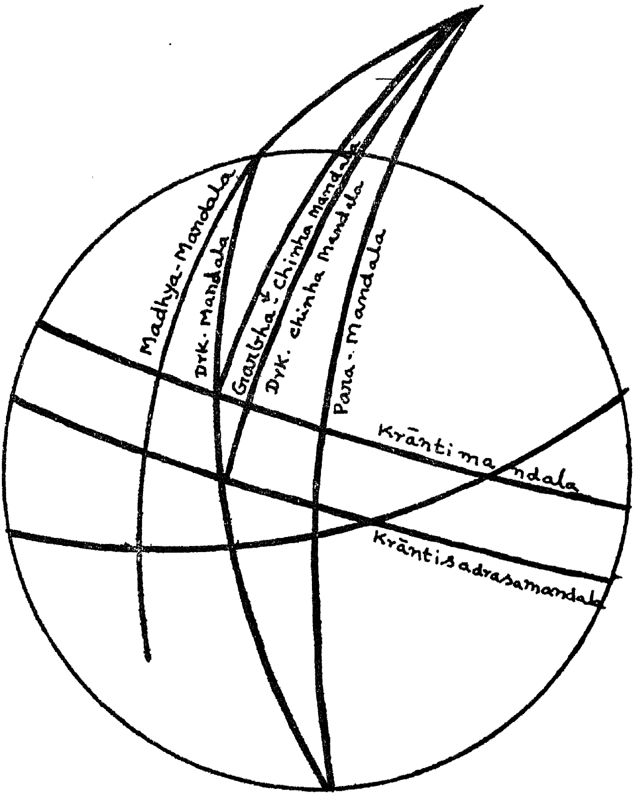


Fig. 99-A

Latter half of verse 14. The purpose of obtaining the parallax in latitude.

The apparent latitude of the Moon is equal to the algebraic sum of his geocentric latitude and the parallax in latitude. From this apparent latitude are to be calculated the Sthiti-khanda and Marda-khanda of the solar eclipse (by the method described in the chapter on lunar eclipse, taking the eclipsing body or grāhaka to be the Moon and the eclipsed or Grāhya to be the Sun).

Comm. The geocentric parallax of the Moon has a double effect on the occurence of a solar eclipse as mentioned before. If the parallax be resolved along the Ecliptic and along a secondary to the Ecliptic, we have respectively the parallax in longitude and that in latitude. The parallax in longitude makes the apparent moment of conjunction at a given place, differ from the moment of the geocentric conjunction, whereas the parallax in latitude makes the magnitude of apparent latitude of the Moon at the place of observation differ from that of the geocentric. The apparent latitude is equal to the sum or difference of the geocentric latitude and the parallax in latitude. Having got the apparent latitude, the computation of the Sthiti and Marda-khandas could be done according to the method described in the chapter on lunar eclipse.

For a point C on the surface of the Earth, a solar eclipse occurs if the latitude of the Moon be less than MD

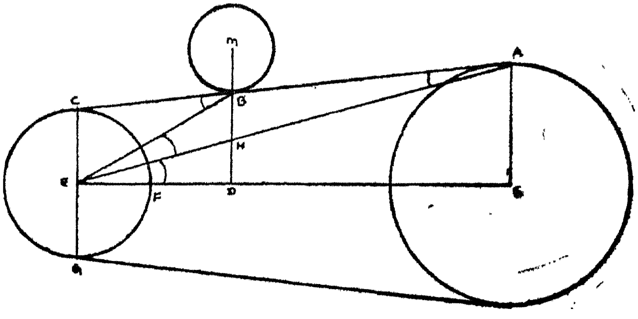


Fig. 100

(vide fig. 100) where M is the centre of the Moon and MD the latitude of the Moon at the point of first contact

$MD = MB + BD = m + \widehat{BED}$ where m is the semi-diameter of the Moon's disc. But

$\widehat{BED} = \widehat{BEA} + \widehat{AES} = \widehat{CBE} - \widehat{CAE} + \widehat{AES} = P - p + s$ where P and p are the parallaxes of the Moon and the Sun and s the angular semi-diameter of the Sun. Thus in order that a solar eclipse may be possible for some point of the Earth, the latitude of the Moon at the moment of conjunction must be less than $P + s + m - p$ $= 57' + 16' + 15' = 89'$ approximately. The lesser the northern latitude of the Moon at the moment of conjunction, more places situated on the surface of the Earth between C and F will have solar eclipse where F is the sub-solar point ie. the point of the Earth which has the Sun in the zenith at the time of conjunction. Similarly, if the southern latitude of the Moon is less than 89′ at the moment of conjunction the places situated on the Earth between G and F will have solar eclipse. In particular the sub-solar point F will have solar eclipse if the latitude of the Moon at the moment of conjunction is less than HD ie. less than $s+m$ ie. 33′ approximately. The sub-solar point will have no parallax, so that the terms P and p in $P+s+m-p$ vanish. For the other points ie. points between F and C or G parallax will be there and the latitude may be greater than $s + m$ but less than ’$-p$ to have an eclipse. A latitude of 33′ corresponds to a distance of $\frac{33 \times 15}{70} = \frac{99}{14} = 7\frac{1}{14}^{\circ}$ of the Sun with respect to a node. Thus if at the moment of conjunction, the latitude of the Moon be less than 7°, even the sub-solar point must have an eclipse.

Verses 15, 16, 17. To find Sparśakāla, Mokṣakāla, Sammīlanakāla and Unmīlanakāla.

First compute the time called Sthiti-khanda (as mentioned in the chapter on lunar eclipses). The ending moment of local Amāvāsyā or what is called the moment of local conjunction is known as the Madhya-Grahakāla or the moment of the middle of the eclipse. Subtract the Sthiti-khanda from the computed time of Geocentric conjunction; the result will be the approximate Spars'a-kāla. This has to be rectified for parallax in longitude as well as the approximate Madhyagrahakāla of geocentric conjunction to obtain the local Spars'akāla and the local Madhyagrahakāla; Similarly the Mokṣakāla, the Sammīlana and the Un-mīlanakālas are to be rectified for parallax in longitude. But while effecting this correction for the parallax in longitude, the Moon's latitude also differs for the corrected time which in turn effects the durations of Sthiti-khanda, Mokṣa-khanda etc. Correcting the first computed Sthiti-khanda. Mokṣa-khanda etc. for this variation in the latitude, and subtracting the Sthiti-khanda from the time of Madhya-graha, we have a better approximation for the Spars'akāla. In as much as parallax in longitude, that in latitude, and the Moon's latitude vary from time to time, and the times of Spars'a, Madhyagraha etc. are effected by them, the process of computation proceeds by the method of successive approximation. Subtracting the rectified Marda-khanda from the rectified Madhyagrahakāla, we have the true Sammīlanakāla; similarly adding the former to the latter we have the true Un-mīlanakāla.

If, as mentioned before in verse 9, the parallax in longitude is found without using the method of successive approximation, the Spars'a-kāla and the Mokṣa-kāla are had at once. But the latitude of the Moon and the parallax in latitude are to be computed using the then longitudes of the Moon and the non-agesimal.

Comm. Clear.

Verses 18, 19. To obtain the true values of the Bhuja and the Iṣṭakāla.

The remaining work proceeds on the lines indicated in the chapter on 'lunar eclipses' (ie. the computation of the Bimbavalana, Bhuja, Koti and the like is to be done as indicated there). The Bhuja will be rectified by multiplying it by the Sthiti-khanda obtained by adopting the latitude of the Moon effected by parallax in latitude and divided by the Sthiti-khanda rectified for parallax in longitude. Similarly given the grāsa ie. the magnitude of the eclipse, the result found before by verse 15 in the chapter of lunar eclipses, is to be multiplied by the Sthiti-khanda rectified for parallax in longitude and divided by that obtained adopting the latitude of the Moon effected by parallax in latitude, and the result so obtained being subtracted from the Sthiti-khanda, we get the Iṣṭa-kāla.

Comm. Refer fig. 73. The Sthitikhanda is the time taken by the centre of the eclipsing body to go from C_1 to N relative to the eclipsed body, ie. keeping the eclipsed body fixed. The Bhuja at any intermediate point of time between the moment of first contact and the middle of the eclipse is NC of fig. 73; and the Iṣṭakāla is the time elapsed between the moment of first contact to the moment when the centre of the eclipsing body occupies any arbitrary position C. In the context of the lunar eclipse, the Bhuja was calculated by the formula (T—I) $(m_1 - s_1)$ where T is the Sthiti-khanda, I = Iṣṭakāla, m_1 and s_1 the motions of the Moon and the Sun on the day concerned. This Bhuja may be also expressed in the form $\sqrt{(R+r-g)^2-\beta^2}$ where R, r are the radii of the eclipsing and eclipsed bodies, of the grāsa and β the latitude of the Moon at the middle of the eclipse. Given the grāsa G, the formula to find I the Iṣṭakāla, is

$$\frac{\quad g)^2-\beta}{m_1-s_1}.$$

In the present context of the solar eclipse, β, the latitude of the Moon is effected by the

parallax in latitude, so that it is a variable. We are to make a correction for this variability, both in the computation of Sthiti-khanda, Bhuja and Iṣta-kāla. The formula for the Sthiti-khanda is $\frac{\sqrt{(R+r)^2-\beta^2}}{m_1-s_1}$ where β is the latitude of the Moon at the moment of conjunction. The value of MN at the moment of conjunction will not be equal to its value at any intermediate point because parallax in latitude differs from position to position of the Moon. In other words β is variable. The formulae given for the rectification of the Bhuja B or the Iṣtakāla I are

$$B' = \frac{B \times T'}{T} \text{ and } I' = T' - \frac{\sqrt{(R+r-g)^2-\beta^2}}{m_1-s_1} \times \frac{T}{T'}$$

T′ is the Sthiti-khanda rectified for the variability of β B′ is the Bhuja rectified for the same whereas T and B are the values of the Sthiti-khanda and Bhuja computed taking the effect of parallax in longitude above.

The effect of parallax in longitude is to prepone or postpone the moment of first contact as well as that of conjunction. The verse under commentary uses two terms Sphuta-Sthiti-khanda and Sphuteshuja-Sthiti-khanda. The former is the Sthiti-khanda rectified for parallax in longitude whereas the latter is that rectified for parallax in latitude ie. by adopting β' instead of β in the formula

$$\frac{\sqrt{R+r^2-\beta^2}}{m_1-s_1} \text{ where } \beta' = \beta \pm \text{effect of parallax in latitude.}$$

Suppose on account of parallax in longitude the moment of first contact t_1 becomes $t_1+\delta t_1$, and let the moment of conjunction t_2 become $t_2+\delta t_2$. Then the Sthitikhanda unrectified for parallax in longitude will be (t_2-t_1) whereas that rectified for parallax will be $(t_2-t_1) + (\delta t_2-\delta t_1)$. This rectified Sthitikhanda is called Sphuta-Sthiti-khanda. Now the verse under commentary gives a procedure to rectify the Bhuja, and Iṣtakāla in the

wake of β being effected by parallax in longitude. formula for Bhuja is $\frac{\sqrt{\overline{R+r-g}^2-\beta^2}}{m_1-s_1}$ wherein all quantities except β may be taken to be constant. Suppose β effected by parallax be comes β'. If $\beta' > \beta$, Bhuja will decrease; also the Sthiti-khanda whose formula is $\frac{\sqrt{(R+r)^2-\beta^2}}{m_1-s_1}$ decreases if $\beta' > \beta$. Hence if T$'$ be the new value of T the Sthiti-khanda, $T' < T$. In other words when $\beta' > \beta$, $B' < B$ and $T' < T$. Also if $\beta' < \beta$, both B$'$ and T$'$ will be greater than B and T respectively. Hence as a rough measure B and T are taken to vary together positively or negatively and as such proportionally. Though both increase or both decrease together, strictly speaking the concept of proportionality is there; but roughly speaking they are taken to vary proportionally which means $B' = \frac{B \times T'}{T}$. The fact that proportionality is not there could be seen in two ways. $B = (T-I)(m_1-s_1)$ (1) with usual notation so that taking B and T to vary on account of the variation in β, $\delta B = \delta T\,(m_1-s_1)$, I not varying so that $B + \delta B = B' = (T+\delta T)\,(m_1-s_1) - I\,(m_1-s_1) = T'(m_1-s_1) - I\,(m_1-s_1) = (T'-I)(m_1-s_1)$ (2).

Dividing (1) by (2) $B/B' = \frac{T-I}{T'-I}$ which will be approximately equal to T/T$'$ provided I is very small compared with T. Assuming so, B/B, could be taken to be equal to T/T, which means $B' = \frac{B \times T'}{T}$ as mentioned in the verse. Or again, considering the formulae for B and T and differentiating them with respect to β and getting B$'$ and T$'$, we shall have the following working.

$\frac{-g)^2-\beta^2}{-s_1}$ so that $2B\delta B =$

or $\delta B = \frac{-\beta\delta\beta}{B(m_1-s_1)}$; similarly $T^2 = \frac{(R+r)^2-\beta^2}{m_1-s_1}$ so that

$2T\,\delta T = \frac{-2\beta\delta\beta}{m}$ so that $\delta T = \frac{\beta\delta\beta}{T(m_1-s_1)}$

$\therefore\ B^1 = B+\delta B = B-\frac{\beta\delta\beta}{B(m_1-s_1)}$ and $T^1 =$

$T - \frac{\beta\delta\beta}{T(m_1-s_1)}$

$\therefore\ \frac{B^1}{T^1} = \frac{B^2-\beta\delta\beta}{B(m_1-s_1)} \times \frac{T(m_1-s_1)}{T^2}$

$T-\frac{\beta\delta\beta}{T}$. Since $\frac{\beta\delta\beta}{B} \pm \frac{\beta\delta\beta}{T}$

$\frac{B^1}{T^1} \pm \frac{B}{T}$.

Regarding the finding of Iṣṭakāla when g the grāsa is given, we have the formula

$T-I = \frac{B}{m_1-s_1} = \frac{\sqrt{(R+r-g)^2-\beta^2}}{m_1-s}$ so that putting $T-I=t$

$\frac{\sqrt{(R+r-g)^2-\beta^2}}{m_1-s_1}$ $\therefore$ As β increases t decreases so that $T-t$ increases ie. I increases. Whereas as β increases T decreases. This means in a way that as T decreases. I increases so that I is taken to be inversely proportional to T ie. I' is taken to be $\frac{I\times T}{T'}$ as given.

Here also, it could be seen that the inverse proportionality is not strictly there; for β increasing both t and T decrease so that $T-t$ ie. I will increase only if the decrease in t is greater that in T. But t^2

and $T^2 = \frac{(R+r)^2-\beta^2}{-s\ ^2}$ so that

Thus we see that as β increases both t and T decrease on account of the minus signs in δt and δT. Also if I were to increase, $\frac{\beta\delta\beta}{t\,(m_1-s_1)^2}$ should be greater than $\frac{\beta\delta\beta}{T\,(m_1-s_1)^2}$ ie. $\frac{1}{t}$ must be greater than $\frac{1}{T}$ ie.

T should be greater than t and this is so. Hence as β increases t and T decrease but $T-t$ increases ie. as T decreases $T-t$ increases ie. I increases. So, roughly I is taken to be inversely proportional to T so that $I' = \frac{I \times T}{T'}$

If strict inverse proportionality is there, since,

and $T + \delta T = T' = T - \frac{\beta\delta\beta}{T\,(m_1-s_1)^2}$ and therefore

$$\frac{t'}{T'} = \frac{t}{T}\frac{\left\{1-\frac{\beta\delta\beta}{t^2\,(m_1-s_1)^2}\right\}}{\left\{1-\frac{\beta\delta\beta}{T^2\,(m_1-s_1)^2}\right\}} = \frac{t}{T}\frac{\left\{t^2 - \frac{}{(\ }\right.}{\left\{T^2 - \frac{}{(m_1-s_1)^2}\right.}$$

t^2 must be roughly equal to T^2 which means $T - t = I$ must be small. In other words the formula of the verse holds good only for small I's or Iṣṭakālas.

In fact this process of rectification of the Bhuja was first given by Brahmagupta in verses 18 and 19 of Sūryagrahaṇādhikāra and accepted by Srīpati in verse 14. The verse 19 under Lunar eclipses in the Sūryasiddhānta also seems to indicate this process but uses the words Madhya-Sthiti-khanda and Spaṣta-Sthiti-khanda in

an ambiguous sense. Ranganātha in his commentary takes the Madhya-Sthiti-khanda to mean that rectified for variation in β, and Spaṣta Sthiti-khanda to mean that rectified for parallax. Some modern traditional commentators, for example pandit Sitha Ramajha commenting on the verse of the Sūryasiddhānta and the commentator of the verses of Brahma Sphutasiddhānta published under the editorship of Acharya Rama Swarupa Sharma, have ignored the variation in β and confined themselves only to the effect of parallax in longitude.

Yet one more proof could be adduced in this behalf. Let D, be the moment of geocentric conjunction, t the time for a given grāsa in between D and T the moment of first geocentric contact. Then the times D, t and T are to be corrected both for parallax in longitude and that in latitude. Take the geocentric Sthiti-khanda, and geocentric Bhuja as the mean-values T and B and those corrected first for parallax in longitude as the True values T' and B'. Also take the time t corrected for parallax in longitude to be t'. We could write $T'-T=\delta T$, $t'-t=\delta t$ and $B'-B=\delta B$. In the above, consider B expressed in time. Then $D-T=S$ the mean Sthiti-khanda, $D-t=B$ the mean Bhuja, $D'-T'=S'$, the true Sthiti-khanda rectified for parallax in longitude, $D'-t'=B'$ the Bhuja also rectified for the same. The $\delta B=\delta D-\delta t$, $\delta I=\delta D \;\; \delta T$. Then using the proportion "If for $D-T$ we have $\delta D-\delta T$ as the variation what shall we have for $D-t$?" The result

$D-T$

$$'-T') = \frac{\text{Mean Bhuja} \times \text{True Sthiti-khanda}}{\text{Mean Sthiti-khanda}}$$

In other words to obtain the Bhuja rectified for parallax in longitude, we have to multiply the mean Bhuja by the Sthiti-khanda rectified for parallax and divide by the Mean Sthiti-khanda. Similarly to obtain the Sthiti-khanda rectified for the variation in latitude also, take the Sthiti khanda rectified for parallax as the mean and that rectified for variation in β as the True and then by the same procedure, Bhuja rectified for parallax in latitude will be equal to Sthiti-khanda rectified for parallax in latitude multiplied by the Bhuja rectified for parallax in longitude divided by the Sthiti-khanda rectified for parallax in longitude Bhāskara gives the names Sphuta Sthiti-khanda and Sphuta S'araja-Sthiti-khanda to the Sthiti-khanda rectified for parallax in longitude and that rectified further for parallax in latitude. The proof adduced above accords with the statements of Brahmagupta, Srīpati and Bhāskara; Ranganātha bearing in mind Bhāskara's version puts a correct interpretation on verse 19, Lunar eclipses of Sūryasiddhānta. But the usage of the words Madhya and Sphuta in that verse, misled the modern traditional scholars including Burgess. If is to be mentioned here that the word Koti translated as perpendicular by Burgess is misinterpreted by him (see his commentary under verse 19, lunar eclipses) as 'The perpendicular is furnished us in time and the rule supposes it to be stated in the form of the interval between the given moment and that of contact or separation'. This translation goes against the definition of Koti contained in verse 18 just above, since the Koti of Sūryasiddhānta is the Bhuja of Bhāskara and vice versa.

We shall now see why there arose confusion in the minds of many modern commentators including Burgess. The problem mooted by the Sūryasiddhānta in calling for a rectification of the Koti, is to be noted as that when the grāsa g is given and not the Iṣṭakāla I. The two formulae

for Koti are $\frac{R+r-g^2-\beta^2}{u-v}$ and $(T-I)(u-v)$. When I is given we have to use the latter formula and take T'' in the place of T where T, T' and T'' are respectively the Sthityardhas (1) Mean, (2) Mean rectified for parallax in latitude and (3) Mean rectified for parallax in longitude. Of course here to arrive at T'', method of successive approximation is to be used as β goes on changing from time to time and there is an inter-play between the simultaneous effects of parallax in longitude and that in latitude.

On the other hand when g is given we have to use the former formula for the Koti and the Koti thus obtained is to be rectified for the variation in β. So Sūryasiddhānta proposes the formula

$\frac{\sqrt{(R+r\ \ g)^2-\beta^2}}{u-v}\times\frac{T'}{T''}$ meaning thereby that the correction for the variation in β is more important because the formula is in terms of β and not the other formula. This formulation is approximate but adopted for the sake of ease. Otherwise from the grāsa, I is to be obtained and the other formula could be used which method is more laborious.

Bhaskīra's Correction of Brahmagupta's Statement

Verses 1, 2 *&* 3. The statement of Brahmagupta namely that the arc of the Moon's Dṛk-kṣepa will be obtained by the sum or difference of that of the Sun with the latitude of the Vitribha, I (Bhāskara) do not accept. I shall give the reason why. In a place where the latitude is 24°. when the longitudes of the Sun, the Moon and the Node are all 180°, at the time of Sun-rise, the ecliptic occupies the position of the prime-vertical. The Moon will not leave the ecliptic even though depressed by parallax in longitude. Thus there is no parallax in

latitude. The Vitribha then being in the zenith, and its latitude being $4\frac{1}{2}°$, the Dṛk-kṣepa of the Moon according to Brahmagupta's formula will be $4\frac{1}{2}°$ and therefore the parallax in latitude obtained by the Dṛk kṣepa will be

$$\frac{790'\text{-}35}{15} \times \frac{\text{H sin } 4\frac{1}{2}°}{3438} = \frac{52'\text{-}42''}{3438} \, 270 = 4'\text{-}8''$$ which is not the case actually.

Comm. Having thus shown the flaw in Brahmagupta's approximate formula, Bhāskara proceeds to show how that formula could be justified in a particular way. Brahmagupta assumed the Moon's orbit to be the ecliptic because at the moment of an eclipse, the latitude of the Moon is very small so that he might be taken to be on the ecliptic. In figure 101 let the Ecliptic coincide with the prime-vertical ZE. Let EV be the Moon's orbit where V is the Vitribha of the Moon's orbit. The arc of the Sun's Dṛk-kṣepa is here zero and that of the Moon's Dṛk-kṣepa is ZV which is the sum of the arc of the Sun's Dṛk kṣepa namely zero and the latitude of the Moon's Vitribha namely ZV, since the pole of the ecliptic now coincides with the south point S, which means that ZV is the latitude of V.

Let VV′ be the nati which is equal to AB, since V′A is the so-called Vikṣepa Sadṛśamandala or the deflected position of VE on account of parallax in latitude. The four minutes of parallax in latitude is now VV′=AB. This parallax is obtained because, Bhāskara argues, the nati obtained by Brahmagupta is there because he took VE to be the Ecliptic and so obtained that nati with respect to VE. Now, Bhāskara says, this is to be corrected by the difference of the Moon's latitudes the original one and that obtained after the Moon is deflected by the parallax in longitude. This difference is 0—AB=BA. Correcting AB with BA, the result is AB+BA=0 so that ultimately

there is no parallax in latitude ie. no nati at all, as should be the case.

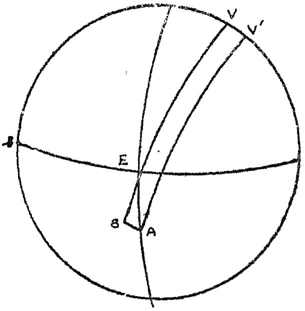


Fig. 101

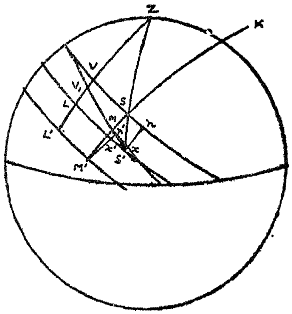


Fig. 102

Or, Bhāskara's argument could be better illustrated from figure 102. Let VS be the ecliptic where V is the Sun's Vitribha and S, the Sun. Let V′M be the Moon's orbit called Vikṣepamandala where V′ is taken to be the Moon's Vitribha and M the Moon. Let S′, M′ be the deflected positions of the Sun and the Moon on account of parallax. ZV′=ZV+VV′ = the arc of the Sun's Dṛk-kṣepa + the latitude at the point V called Vitribhalagna—Bāna, as formulated by Brahmagupta that ZV′ is roughly equal to the Moon's Dṛk-kṣepa-Dhanus. (Strictly speaking ZV′ ought to be perpendicular to the Moon's orbit V′M, if V′ were to be the Vitribha of the Moon; but, as the latitude is small, the error is negligible). Let LS′ and L′M′ be the so-called Krānti-Sadṛś'a-mandala and Vikṣepa Sadṛś'amandala or parallel drawn to the ecliptic and the Moon's orbit through the deflected positions S′ and M′ of the Sun and the Moon. S′n is the Sun's parallax in latitude ie. Nati. Similarly Brahmagupta took M′n' to be the nati of the Moon as stated by Bhāskara. The error committed will be therefore M′n'—S′n = (M′x'+$x'n'$)—(S′x+xn)=M′x' − xn) cancelling $x'n'$ and S′n which are roughly equal. Also xn could be roughly taken to be equal

to MS. Hence the relative parallax is equal to $M'x' - MS$ = Difference of the latitudes of the Moon in his original and deflected positions respectively. Hence $S'n = M'n' - (M'x' - MS)$ = Brahmagupta's nati+correction of the difference of the latitudes reversely effected, as stated by Bhāskara.

In fact, Bhāskara has misread Brahmagupta's correct procedure, since the latter sought the relative parallax of the Sun and the Moon.

Instead of adding the latitude at V to the zenith-distance of V (the arc of the Sun's Dṛk-kṣepa) which implies additional computation of that latitude, as an approximate procedure, MS is added to ZV to get ZV' as an alternate procedure as mentioned by Bhāskara in the course of the commentary.

Whereas Brahmagupta was seeking relative parallax, Bhāskara misread that Brahmagupta took $M'n'$ as the nati and did not effect the correction of $(M'x - MS)$ to obtain $S'n$, which Bhāskara took to be the Sphuta-nati. Not effecting the above correction is interpreted as neglecting it since the Moon's latitude during the course of an eclipse is small.

GRAHACCHĀYĀDHIKĀRA

Verse 1. The orbital inclinations of the planets.

The inclinations of the orbits of Mars, Mercury, Jupiter, Venus and Saturn to the ecliptic are respectively 110, 152, 76, 136 and 130 minutes of arc. The nodes of Venus and Mercury get rectified by adding their respective S'īghra anomalies to their values obtained originally.

Comm. The values given above are said to be the mean values. Those given for the superior planets namely Mars, Jupiter and Saturn approximately accord with their modern values. Bhāskara says that these values pertain to that moment of observation, when the S'īghra anomaly is equal to $90+\frac{1}{2}$ H $\sin^{-1} a$ where a is Hsine of the maximum S'īghraphala. This is quite in order because when the S'īghra anomaly assumes the said value, the true planet is at the point of intersection of the deferent and the eccentric, which means that the planet is equidistant from E_1 as well as E_2 (vide fig. 103). Identifying E_1 to be the earth's centre and E_2 to be that of the Sun in the case of the superior planets, the mean latitude of the planet observed will be the same as that observed either from the earth or from the Sun. Hence in the case of the superior planets, the maximum latitudes of the planets observed accord with the geocentric as well as heliocentric observation. These are taken to be the mean values of the maximum latitudes.

In the case of the inferior planets, it will be clear why the modern values of 7° and 3°–24′ for Mercury and Venus are far higher than the Hindu values namely 152′ and 136′. Since the mean planet in this case is taken to be the Sun, the linear values of the latitude observed from E and S, the centres of the Earth and the Sun respectively will be in the ratio SP/ES (vide fig. 104).

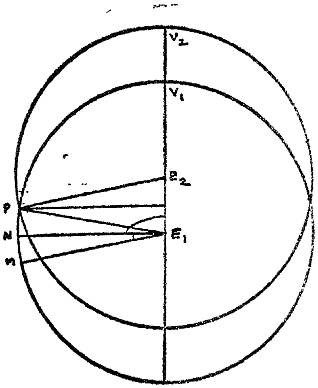


Fig. 103

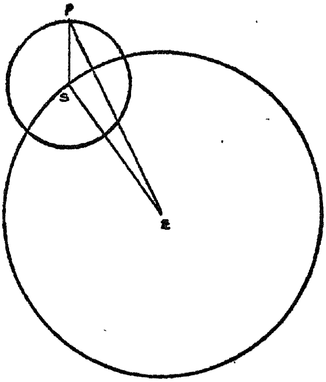


Fig. 104

In the case of Mercury this ratio is and that in the case of Venus is $\frac{7}{10}$. Hence the values 420×4 and $\frac{204\times7}{10}$ ie. 168′ and 142′ roughly accord with the Hindu values. In other words the Hindu values are geocentric and the modern heliocentric.

Bhāskara adds that the nodes of Mercury and Venus as computed previously are to be increased by the S'īghra anomaly to obtain the actual longitude of the node from which the latitude is to be computed. This directive is a beautiful example to show implied heliocentric motion. Let the convex angle ASN be the heliocentric longitude of the node measured negatively as the node has a negative motion along the ecliptic. Adding the heliocentric S'īghra anomaly to this longitude of the node means convex angle $ASN+\widehat{USP}=360^\circ+\widehat{NSP}-\widehat{ASU}=\widehat{NSP}-\widehat{ASU}$. Now add the longitude of the planet ie. here $\widehat{ASU}$ $(=\widehat{AES})$ to obtain the argument from which the latitude of P the planet is to be computed. The result is NSP as should be.

Computing the latitude as indicated in the next verse, by the formula $\frac{\text{H sin NSP}\times\beta\times\text{R}}{\text{R}\times\text{R}}$ where β is the maximum latitude cited above in the respective cases, multiplied by *R* and divided by K indicates that the linear magnitude of the latitude will be increased or decreased according as K is smaller or greater than R.

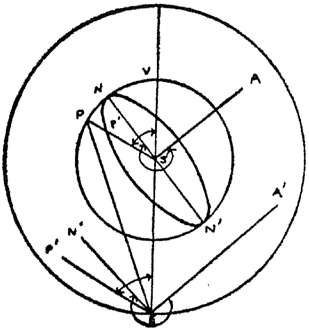


Fig. 105

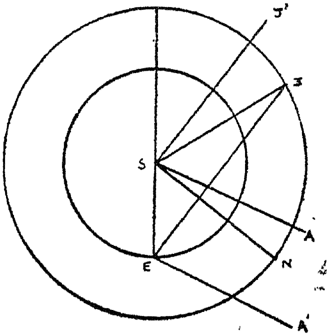


Fig. 106

(a) In this context, we are to throw light on some moot points. Bhāskara says "मन्दस्फुटो ग्रहः स्वशीघ्रप्रतिमण्डले ', which tantamounts to saying that the node is situated in the heliocentric orbit "स्वशीघ्रप्रतिमण्डल". The Vimandala or the planet's orbit, taking for example the inner orbit of P in fig. 105, does not exactly lie in the plane of the ecliptic as shown in that figure but is in an inclined position cutting the coplanar inner orbit in N and N′, as shown by the ellipse NPN′ where NP=NP′. The argument to calculate the latitude is NP′ and the formula is $\frac{\text{H sin NP}'}{\text{R}}\times\beta$ to give the heliocentric linear latitude. To obtain its geocentric linear value, we have to multiply by $\frac{\text{R}}{}$

(b) In the case of the superior planets, Bhāskara mentions in the Golādhyāya "पातोऽथवा शीघ्रफलं विलोमं कृत्वा स्फुटात् तेन युतात शरोऽत:" verse 22 (Golabandhādhikāra). This also clearly indicates that the orbits of the superior planets are also heliocentric, for, otherwise, there is no purpose of subtracting the S'īghraphala from the True position of the planet. The purpose in doing so is to obtain the heliocentric longitude of the planet (vide fig. 106).

True geocentric longitude of J is $A'EJ = ASJ'$.

Subtracting the S'īghraphala $EJS = JEJ'$ from ASJ', we have $\widehat{ASJ}$ the heliocentric longitude. Adding the retrograde longitude of N (तेन युतात् as cited above in verse 22) means adding ASN to $\widehat{ASJ}$ which gives $\widehat{NSJ}$. This is the argument to calculate the latitude of J.

(c) Bhāskara alludes to a confusion in the mind of Chaturvedāchārya in this context while commenting upon verses 23, 24 in Golabandhādhikāra (Golādhyāya). Chaturvedācharya commenting on Brahmagupta's words (verse 10 Grahayukti-adikāra ch. 9, Brahma Sphutasiddhānta) exclaims. "The latitude of Mercury and Venus will be the same as what they are at the point of S'īghroccha; correctness of the result alone is proof; no other reason could be adduced!" Bhāskara clears his misconception in the following words. "The number of sidereal revolutions of the nodes of Mercury and Venus mentioned in the chapter on mean motion, are to be increased by the number of the sidereal revolutions of the S'īghrocchas of Mercury and Venus, as mentioned by Mādhava in his work "Siddhānta-chudāmani". This means that the S'īghra anomaly is to be added to the position of the node obtained by the smaller number of revolutions given in the chapter on mean motion.

The misconception in the mind of Chaturvedāchārya as well as the wrong notion in the minds of Mādhava and even Bhāskara in construing that the number of revolutions of the nodes are to be increased by the number of revolutions of the S'īghrocchas, are due to the fact that it was overlooked that the position of S'īghrocchas with respect to Mercury and Venus are given by their heliocentric longitudes. In other words as is mentioned by the verse "ज्ञशुक्रयोः ग्रहः सूर्यः भवेत्तौ शीघ्रनामकौ" the S'īghrocchas of Mercury and Venus are no other than the heliocentric positions of those planets. Thus the exclamation of Chaturvedāchārya cited above arose out of thinking that the *S'īghrocchas differ from the planets*, whereas they are the same heliocentrically, though they differ geocentrically. Heliocentrically the planets longitude is (vide fig. 105) $\widehat{ASP}$ which is equal to $A'EP'$ geocentrically. The geocentric longitude of the planet is on the other hand $A'EP$ differing from the above, though both the heliocentric and geocentric longitudes point to the same planet. Bhāskara, no doubt, gave a correct procedure but missed to identify the S'īghroccha and the heliocentric position of the planet. In this context, the reader is referred to the author's 'peculiar concept of S'īghroccha in Hindu astronomy published in the journal of Oriental Research of the S. V. University Vol. XIV, Part 2, Dec. 1971.

Verse 2. The Hsine of the arc of the planet's orbit Vimandala intercepted between the nearer node and the planet multiplied by the maximum latitude of the planet cited, and divided by the S'īghrakarṇa gives the latitude of the planet at the given place.

Comm. From the formula akin to that which gives the declination of a point on the ecliptic namely $\sin \delta = \sin \lambda \sin \omega$ or in the Hindu form $H \sin \delta = \dfrac{H \sin \lambda \; H \sin \omega}{R}$,

$\text{H} \sin \beta = \frac{\text{H} \sin \lambda \; \text{H} \sin i}{\text{R}}$ where β is the latitude required, i the maximum latitude and λ the arc of the planetary orbit intercepted between the nearer node and the planet. Since β and i are small H sin β and H sin i could be taken to be β and i. Hence we have $\beta = \frac{i \times \text{H} \sin \lambda}{\text{R}}$ Since this value is had at the distance of the S'ighrakarṇa, its value at a distance of R should be given by

$$\beta = \frac{i \, \text{H} \sin \lambda}{\text{R}} \times \frac{\text{R}}{\text{K}} = \frac{i \, \text{H} \sin \lambda}{\text{R}}$$ as formulated.

3. $\sqrt{\text{R}^2 - \text{H} \sin^2 v}$ is called Yaṣṭi where v is Āyanavalana. The latitude β of the planet multiplied by Yaṣṭi and divided by R or multiplied by H cos δ where δ is the declination of the point whose longitude is $90 + \lambda$, λ being that of the planet and divided by R gives the value of the rectified latitude which could be added to the declination of the foot of the latitude to give what we say the modern declination of the planet.

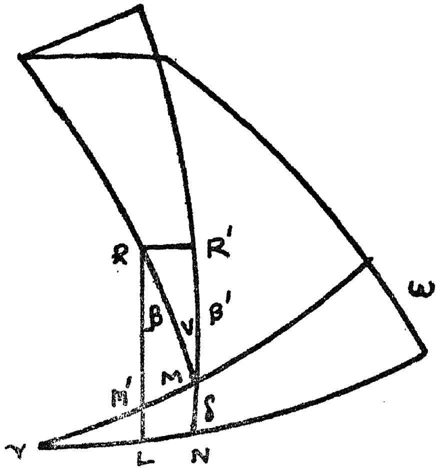


Fig. 107

Comm. Let rM (fig. 107) be the ecliptic and rN the celestial equator whose poles are K and P respectively. Let R be a celestial body whose latitude is β and whose modern declination is RL. Let M be the foot of the latitude circle and let δ be the declination of M.' The word Krānti in Hindu astronomy is applied to connote the declination of a point on the ecliptic alone and not of any other point like R. RL is called *Sphuta Krānti* which is equal to $R'N = R'M + MN$. RM is called *Vikṣēpa* and

Sphuta Vikṣēpa. KMP is called the Āyanavalana at the point M of the ecliptic. Produce MP to P' where $PP' = \delta$ so that $MK = MP = 90°$. Hence $\hat{P'}$ ~~is a~~ right angle and $KP' = K\hat{M}P' = v =$ Āyanavalana. From the spherical triangle KPP', $\cos \omega = \cos v \cos \delta$. Draw perpendicular RR' on MP so that $MR' = \beta' =$ Sphuta Vikṣēpa $= \beta \cos v$

But H cos v =
R

$$\therefore \; \beta' = \frac{\text{Yaṣti} \times \beta}{R}$$ which is to be added to δ to obtain the Sphutakrānti $R'N$ or RL, the modern declination.

Note (1) One Mukhopādhyāya, in his thesis 'The Hindu nakṣatras' submitted to the Calcutta University, mistook RM' to be the Sphutavikṣēpa instead of $R'M$ and hastily remarked that Bhāskara was wrong in making RM' less than RM.

Note (2) $\hat{v}$ shown in the fig. 107, is called Sthanīya-valana or valana at the point M which is considered to be place of the planet 'Sthāna' on the ecliptic. $K\hat{R}A$, on the other hand is called Bimbīya-valana or valana at the bimba or disc of the planet.

Note (3) $\hat{v}$ could be obtained from the spherical triangle KMP where $KM = 90°$, $PM = 90 - \delta$, using the

formula $\cos \omega = \sin (90-\delta) \cos v$ or in the Hindu form $H \cos v = \frac{R\ H \cos \omega}{H \cos \delta}$. In this case $\beta' = \frac{\beta\ H \cos \omega}{H \cos}$

Or again noting that $\widehat{MKP} = 90 - \lambda$ where λ is the longitude of R, we could use 'Inner side Inner angle formula' with respect to the triangle KMP, which gives $0 = \sin 90 \cot \delta - \sin (90 - \lambda) \cot v$ or $\cot v = \frac{\cot \omega}{\cos \lambda}$

or $\tan v = \cos \lambda \tan \omega$. But this formula implies the tangent functions which were not used by the Hindu astronomers.

Similarly using the elements $\overline{90-\lambda}$, 90°, v, and $90-\delta$ of the same triangle KMP, another formula could be got for v. Or again noting that KPM$=90+\alpha$, we could yet get more formulae where α is the Right ascention of M.

Note (4) Thus far we have used modern formulae. Let us now see as to how Bhāskara derives his formula. He takes the triangle MKP′ (fig. 107) wherein MK$=90°$, $' = v$ and $\widehat{P'} = 90$. From fig. 108, the Hsine of MK is

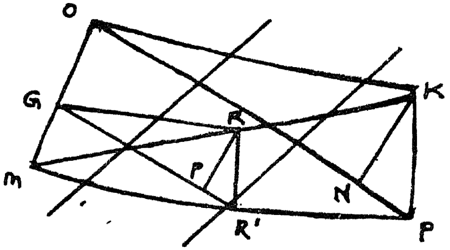


Fig. 108

KO equal to R where 'O' is the centre of the sphere and the Hsine of KP is KN so that it is the Āyanavalanajyā, Hence

is the centre of the sphere and the Hsine of KP is KN so that it is the Āyanavalanajyā. Hence

$^{2}=OK^2-KN^2 \quad \therefore\ ON=\text{Yaṣṭi}=\sqrt{R^2-\text{Āyanavalanajyā}}$ $=$ Āyanavalanakoṭijyā. From the similarity of the triangles OKN and GRF,

$$\frac{OK}{ON}=\frac{GR}{GF} \quad \therefore\ GF=\frac{GR\times ON}{OK}=\frac{H\sin MR\times \text{Yaṣṭi}}{R}$$

GR could be taken equal to MR, the latter being small and GF could be taken to be equal to GR′ ie. H sin MR′ and so equal to MR′.

$$\therefore\quad MR'=\frac{MR\times \text{Yaṣṭi}}{R}$$

Adding MR′ to the declination of M, we get the modern declination of R′ ie. the Sphuṭakrānti of R.

Note (5) In fact the spherical triangle MKP is just like the spherical triangle rE♋ (fig.). In the place of the paramakrānti E♋, we have the Āyanavalana KP (fig. 108), and in the place of Dyujyā ♋L (fig. 19) we have Yaṣṭi.

Note (6) An alternative is given in the verse for this namely $MR'=\dfrac{MR\times H\cos\delta'}{R}$ where δ' is the nation of a point whose longitude is $\overline{90+\lambda}$. In words we have to prove that $H\cos v=H\cos\delta'$ or $v=\delta'$ (of a point whose longitude is $90+\lambda$. Since declination is given by the formula $H\sin\delta=\dfrac{H\sin\lambda\times H\sin\omega}{R}$

Putting $90+\lambda$ for λ, $H\sin\delta'=\dfrac{H\cos\lambda\times H\sin\omega}{R}$

But from triangle MKP, $\dfrac{\sin v}{\sin\omega}=\dfrac{\sin 90-\lambda}{\sin 90-\delta}=\dfrac{\cos\lambda}{\cos\delta}$

$$\therefore \quad \sin v = \frac{\sin \omega \cos \lambda}{\cos \delta} \text{ or } \text{H} \sin v = \frac{\text{H} \sin \omega \times \text{H} \cos \lambda}{\text{H} \cos \delta} \quad \text{II}$$

Comparing I and II, v would be equal to δ' provided H cos δ=R. This is accepted as an approximation, as δ is generally small.

This approximate formula is given by Sūryasiddhānta in the context of Āyana Dṛk-karma in verse 10, ch. 7, where v is assumed to be equal to δ' defined above.

Verses 4 and 6. The process called Āyana Dṛk-karma.

The Āyana Dṛk-karma correction measured in minutes is obtained by multiplying Āyanavalana by the unrectified celestial latitude, and divided by H cos δ and then multiplied by 1800 and divided by the rising time of the Rāśi which is occupied by the planet. Or again it is obtained approximately by the product of the Āyanavalana and the unrectified celestial latitude divided by the Yaṣṭi.

This Āyana Dṛk-karma correction is negative if the hemisphere and the latitude have the same direction.

On symbols, Āyana Dṛk-karma correction =

$$\frac{\text{Āyanavalana} \times \beta \times 1800}{\text{H} \cos \delta \times \text{T}} \text{ or approximately equal to}$$

$$\frac{\beta \times \text{Āyanavalana}}{\text{Yaṣṭi} \ (= \text{H} \cos v)}.$$

Comm. Let G be the position of the planet when the foot of the planet's secondary (called Grahasthāna) to the Ecliptic namely A is rising, where rCA is the Ecliptic. Let rML be the celestial Equator. Let P and K be the poles of the celestial Equator and Ecliptic respectively. The arc AC of the Ecliptic intercepted between A and the declination circle of G expressed in minutes represents the आयनकलाः formulated in the verse. To find the magnitude of AC, the first formula mentioned above envisages finding

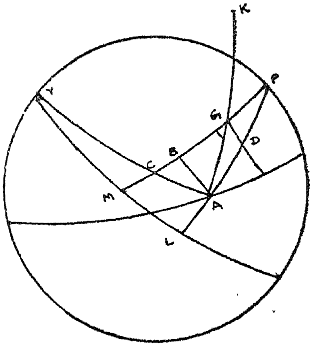


Fig. 109

it through the magnitude of ML the corresponding Equatorial arc which is itself found through finding the magnitude of AB. AB expressed in minutes will be equal to the number of asus taken by the declination circle of G namely PGB to have traversed from the position PDA to PGB which is the time taken by the planet G to be at its position from the moment of its rising at the Equatorial horizon namely PDA, D being then its position. When G was at D, K, the pole of the Ecliptic should have been itself rising on the Equatorial horizon. Thus it is sought to find the time expressed in asus taken by K from its moment of rising on the Equatorial horizon to its present position at K. When Bhāskara mentions in the commentary under the verse that "

by the word क्षितिज, he had at the back of his mind

ie. Equatorial horizon and not the horizon of the place shown in the figure. Also as was mentioned before in a previous context the time expressed in asus taken by an equatorial arc to rise is equal to the number of minutes in that arc. Since the time taken by the planet to traverse the arc of the diurnal circle namely DG is the same as the time taken by the equatorial arc LM to rise, Bhāskara seeks to find the number of minutes in AB, then compute, the number of Asus taken by LM to rise

through which the number of minutes in AC could be computed.

$$AB = AG \sin \widehat{AGB} =$$

$$\frac{AG \times H \sin \widehat{AGB}}{R} = \frac{\beta \times \bar{A}\text{yanavalanajy}\bar{a}}{R} \text{ where } AG = \beta.$$

In the formula given the word Āyanavalana is used for Āyanavalanajyā ie. the Hindu sine of Āyanavalana for brevity. Also, since generally the angle $\widehat{AGB}$ defined as Āyanavalana happens to be small, its Hsine will be almost equal to it. In this sense also the term Valana is used where Valanajyā is to be used.

From AB we pass on to the magnitude of LM. From Hindu spherical Trigonometry

$$LM = \frac{AB \times R}{H \cos AL} = \frac{AB \times R}{H \cos \delta} = \frac{\beta \times \bar{A}\text{yanavalanajy}\bar{a}}{R} \times \frac{}{H \cos \delta} = \frac{\beta \times \bar{A}\text{yanavalanajy}\bar{a}}{H \cos \delta}.$$

(H cos δ is called Dyujyā which is connoted by the term Dyu-guna in the verse, the words Jyā and Guna being synonymous).

$$\text{Thus } LM = \frac{\beta \times \bar{A}\text{yanavalanajy}\bar{a}}{\text{Dyu-guna}}$$

From LM we pass on to the corresponding arc of the Ecliptic namely AC, which does not imply the latitude of the place. So we have to use what are called Niraksha-Udayas of Rāśis ie. the rising times of the Rāśis at Equatorial places. Rule of three is used here to find AC from LM. Locating the particular Rāśi in which the planet is situated, and using the proportion. "If a Rāśi of 30° ie. 1800′ of the Ecliptic rises at an Equatorial place in say x asus, what arc in minutes corresponds to the number of minutes or what is the same the number of Asus

in the arc LM?" We get the magnitude of the arc AC in minutes.

The computation of this arc AC is intended to know the difference between the times of rising of the point A and the point G the former being the point of the ecliptic signifying the position of the planet and the latter being the planet itself. This difference of the times of rising is itself required to know the actual time of rising of the planet by a knowledge of the time of rising of the point which signifies the planet's position on the Ecliptic by its longitude.

In verse 5, Bhāskara gives an alternate and easy method of obtaining the magnitude of AC construing AGC to be roughly a plane triangle which does not involve any appreciable error.

$$AC = AG \tan \widehat{AGC} = \frac{\beta \times H \sin AGC}{H \cos \widehat{AGC}}$$

$$= \frac{\beta \times \bar{A}\text{yanavalanajy}\bar{a}}{\text{Ya}\d{s}\text{ti}}$$ since Yaṣti is defined as H cos AGC in verse 3.

Note. The point C is called the Kṛta-Āyana-Dṛk-Karma Sthāna ie. the point of the Ecliptic signifying the planetary position rectified for the Samskāra or correction called Āyana-Dṛk-karma. The longitude of C thus got is called the polar longitude of the planet which is defined as the longitude of the planet as measured on the Ecliptic upto the point of intersection of the planet's declination circle with the Ecliptic. Bhāskara mentions elsewhere,

ध्रुव

ie. in the case of stars which are fixed, the Sphuta-śaras or polar latitudes (given by GC in the above case)

given and longitudes rectified for Āyana Dṛk-karma are given ie. polar longitudes. Bhāskara made this as an approximate statement, for, we know, even in the case of stars, though they be fixed, their polar latitudes also do change by the precession of equinoxes be it just a little; whereas their polar longitudes do not change by the same constant quantity as their celestial longitudes.

Verses 6, 7 and 8. Obtain the Carakhandas or ascentional differences of the Sphuta and Asphuta Krāntis. If they be of the same direction their difference is to be taken, if of opposite direction, their sum is to be taken. The result in asus gives the Ākṣa Dṛk-karma correction if the celestial latitude is of appreciable magnitude. If it be not appreciably large, it (the celestial latitude) is to be multiplied by the Ākṣavalana, then divid by H cos ϕ or what is the same, multiplied by the equinoctial shadow and divided by 12. The result is to be multiplied by the radius and divided by H cos δ. Then we have the Ākṣa Dṛk-karma correction in asus. Assuming the planet's position corrected for Āyana Dṛk-karma to be the Sun, obtain the lagna using the asus of the Ākṣa-Dṛk-karma. If the planet has a southern latitude let the lagna be found in the positive direction; otherwise in the negative direction. Then we have the rising lagna of the planet or what is the same the longitude of the rising point of the planet. Again assuming the planet's position increased by 180° to be the Sun, and using the asus of Ākṣa Dṛk-karma, obtain the lagna in the positive direction with respect to a northern latitude of the planet or else in the negative direction. The result gives the longitude of the setting planet or what is called the Asta-lagna.

Comm. (Refer to fig. 110) Let p be the planet whose Graha-sthāna or foot of the latitude is p'. Let q be what is called the Kṛta-Āyana-Dṛk-Karmaka-Sthāna or the planet's position corrected for the correction of Āyana

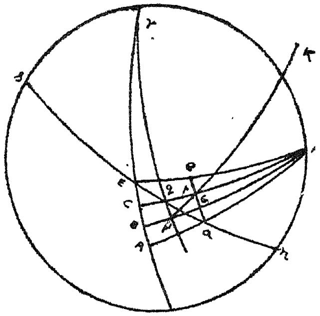


Fig. 110

Dṛk-karma. (This correction is given by the arc $p'q$). The difference of the rising times of the planet p and q expressed in asus is what is sought here. The planet p rose at a and has covered the arc ap of the diurnal circle after rising. So, we have to compute ap and therefrom the corresponding arc of the Equator namely AC. If the planet has no latitude ie. is situated on the ecliptic at p', the arc $p'q$ or Āyana Dṛk-karma vanishes, for, the Āyana Dṛk-karma is no other than the projection of the latitude of the planet on the ecliptic taking P or celestial pole as the vertex of projection. Also if there were no Akṣa ie. if the place be equatorial, p and q rise simultaneously ie. the planet will rise along with the point called Kṛta-Āyana-Dṛk-Karmaka-Sthāna q, mentioned above. In fact the Dṛk-karma corrections Āyana and Ākṣa are contemplated to obtain the difference in the rising times of p and p' ie. the planet and its position on the ecliptic. This time is resolved into two parts (i) the difference of the rising times of p' and q and (ii) that of the rising times of q and p. The former is given by the equatorial arc BC which goes by the name Āyana-Dṛk-karma and the latter by the arc AC which goes by the name Ākṣa-Dṛk-karma. The algebraic sum of these two corrections gives the difference of the rising time of p and p' ie. the time given by the arc AB. Here AB=AC−BC, In verses

4, 5 we have seen how BC is computed. Here we are seeking to compute AC through the corresponding arc ap of the diurnal circle.

The verse uses the word Sphuta-Krānti, Asphuta-Krānti, and their Cara-khandas. Here in the figure pC is the Sphuta-Krānti and $p'B$ is the Asphuta-Krānti or simply Krānti. Also pp' is called Asphuta-Vikṣēpa and bp' Sphuta-Vikṣēpa. Sphuta-Krānti $=pc=bB=bp'+p'B=$ Sphuta-Vikṣēpa + Asphuta-Krānti. In other words the declination of p, the planet, is called Sphuta-Krānti and that of its longitudinal position on the ecliptic namely p' is called Asphuta-Krānti or simply Krānti. We have obtained Sphuta-Krānti from Krānti by adding to the latter the Sphuta-Vikṣēpa. The method of obtaining Sphuta-Vikṣēpa from the Vikṣēpa, otherwise called Vikṣēpa Sphuta-karama was dealt with in verse 3 above.

Now we have to define the Cara-khandas pertaining to the Sphuta-Krānti pC and the Asphuta-Krānti $p'B$. The former is defined as AE and the latter by CE very approximately. We say very approximately because when p' comes to the horizon, the Cara-khanda will not be exactly EC but a little more or less than EC since $Bp' \neq Cq$. In other words, the arc of the equator between the declination circle of q while rising and the equatorial horizon ie. AE is defined as the Cara-khanda of pC; similarly CE is that of $p'B$. Their difference is $AE-CE=AC$. If we use the modern notation, we have to take their algebraical difference. If that be done, the alternative case of adding the Cara-khandas when the Sphuta and the Asphuta-Krānti are of opposite directions, will be automatically implied. The asus pertaining to this arc AC will be the correction of Ākṣa-Dṛk-karma as mentioned in verse (6) when the latitude of the planet is appreciable. These asus will be equal to the time taken by p the planet to cover the arc ap or what is the same, the

time in between the moment of rising of p and that of q, the position of the Kṛta-Āyana-Dṛk-Karmakagraha.

When the latitude is very small, an approximate estimate of this correction is given in asus as

$$\frac{\sin \xi}{\text{H} \cos \phi} \times \frac{\text{R}}{\text{H} \cos \delta} \text{ or } \frac{\beta' \times \text{H} \sin \phi}{\text{H} \cos \phi} \times \frac{\text{R}}{\text{H} \cos \delta}.$$

That this formula is ($\beta' = qp$) only approximate is seen from the fact that H sin $\xi \neq$ H sin ϕ. This second formula can be proved as follows.

Ākṣavalana in asus $= \dfrac{\beta' \times \text{H} \sin \phi}{\text{H} \cos \phi} \times \dfrac{\text{R}}{\text{H} \cos \delta}$ where β' is the Sphuta-Vikṣēpa. Taking qpa as a plane triangle

$pa = qp \tan \widehat{pqa} = qp \times \tan \widehat{\text{PE}n}$ approximately

$= \text{Sphuta-Vikṣepa} \times \dfrac{\text{H} \sin \phi}{\text{H} \cos \phi}$. Hence

$\text{AC} = \dfrac{\beta' \times \text{H} \sin \phi}{\text{H} \cos \phi} \times \dfrac{\text{R}}{\text{H} \cos \delta}$. With respect to the first formula ie.

$$\text{Akṣavalana in asus} = \frac{\beta' \times \text{H} \sin \xi}{\text{H} \cos \phi} \times \frac{\text{R}}{\text{H} \cos \delta},$$

$pa = qp \tan \widehat{pqa} = \dfrac{[illegible]}{\text{H} \cos \widehat{pqa}}$

H cos $\widehat{pqa}$ which is Yaṣti previously defined, H cos ie. H cos ϕ, an approximate value is substituted, because the latitude is small. Having obtained the value of pa, it is then reduced to the Equator by multiplying by R and dividing by H cos δ. The convention with respect to the sign is clear. The idea of Kramalagna and Vilōmalagna may be elucidated as follows. In fig. 110, q, is on the horizon rising whereas p had already arisen. So to

obtain the rising time of the planet p, which occured before that of q, Vilōmalagna or an anterior point of time is to be computed, and this is done by subtracting the asus of Akṣa-Dṛk-karma from the rising time of q. The rest follows on these lines of elucidation.

Verse 9. The planet rises at the time known as its Udayalagna (which is to be computed as aforesaid) and it sets at the time known as its Astalagna (which is also to be computed as mentioned before).

Comm. Clear.

Verse. 10. During the night, at a given moment, the computed Udayalagna of the planet is less than the particular lagna of the moment, ie. if the longitude of the Udayalagna is less than that of the current lagna, and also if the Astalagna of the planet computed is greater than that of the then current lagna ie. if the longitude of the Astalagna is greater than that of the current lagna, the planet is visible ie. above the horizon. In the case of the Moon, however, if he is not eclipsed by the rays of the Sun, he may be visible even shortly after Sunrise or a little before Sunset in contradistinction to a planet which could not be seen at all during day time (except perhaps Venus). When a planet is visible, his gnomonic shadow could be computed. (This does not mean that the gnomon casts a shadow of the planet but means that its zenith-distance could be computed according to the methods described in Trīpraśnādhikāra.

Comm. Clear.

Verse 11. To obtain the shadow, the time that has elapsed after the rise of the planet is to be known.

If it be required to find the gnomonic shadow of the planet, then the current lagna and the Udayalagna of the

planet *at the moment* are to be computed. The time in between the two lagnas, which will be in Sāvana measure pertaining to the planet gives the time that has elapsed after the rise of the planet.

Comm. According to computation, the difference of times of the current lagna and the Udayalagna as computed taking the present position of the planet will be the time measured along the diurnal path of the planet and as such is in Sānava measure.

Bhāskara mentions here a very subtle point. In the analysis the latitude of the planet is taken into account while finding the Udayalagna. Time measured in this case is called Kṣetrātmika and not Kālātmika for the following reason. Suppose A is the Udayalagna of the planet and B the current lagna both points being on the ecliptic. Let us say that the current lagna is posterior to the Udayalagna (there is no loss of generality in supposing this). By the time of the current lagna, the planet is no more at A but will have moved a little towards B. Let the present position of the planet on the ecliptic be A′. The corresponding arc of AB on the equator gives the sidereal measure of the time that has elapsed after the rise of the planet whereas the corresponding arc of A′B on the equator gives the Sāvana measure of the same time. Here this Sāvana is that pertaining to the planet and not that pertaining to the Sun which is Saura Sāvana. In other words it is the Sāvana pertaining to the particular planet, because the arc AA′ is traversed by the planet in question as per its own velocity.

The stress is here on the word Tat-Kāla-graha, ie. the longitude of the planet at the moment at which the shadow or the zenith-distance of the planet is sought. Taking this as the position of the planet and taking the latitude of the planet also into account compute the Udayalagna. This will be A′ cited above. To obtain the

sidereal measure of the same time we have to take A and not A′.

Verse 12. Sāvana measure alone is to be employed while finding the gnomonic shadow ie. while the zenith-distance of the planet is to be computed, because the arc of the diurnal circle of the planet indicates only Sāvana measure. Suppose the Udayalagna falls short of the current lagna, then the Bhōgyakāla ie. the remaining rising time of the Rās'i in which the planet is situated added to the elapsed time of the Rās'i of the current lagna together with the sum of the rising times of the Rās'is in between gives the difference of the Udayalagna of the planet and the current lagna.

Comm. (Ref. fig. 111). Let *p* be the planet's place on the Ecliptic at the moment in question when the lagna is L (or the foot of the latitude of the planet in case it is not on the ecliptic). Let PA be the remainder of the Rās'i in which the planet is situated. Then the time of rising of the arc PA is here termed Bhōgyakāla. Let DL be the arc of the Rās'i in which the current lagna L is situated, which has arisen. The time of rising of this arc DL is called Bhuktakāla. The times of risings of the Rās'is in between namely AB, BC, CD are called Madhyōdayās. The sum of the rising times of PA, AB, BC, CD, DL gives the time in between the rise of P and that of L which is the time that has elapsed after the planet has arisen upto the current Lagna ie. the rising time of L.

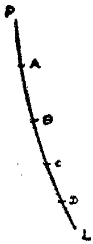


Fig. 111

Verse 13. The method of computing the gnomonic shadow of the planet or what is the same the zenith-distance of the planet, as per the method of computing that of the Sun (extended to the case of the planet) after

finding the Sphutakrānti (ie. modern declination) of the planet.

The Krānti of the planet or the declination of the foot of the latitude of the planet added to the latitude rectified called Sphutasāra, gives what is called Spaṣtakrānti of the planet and its Hsine is called Spaṣta-Krāntijyā. From this H sin δ, H cos δ etc. is to be computed (as mentioned in Trīpras'nādhikāra in the context of finding the zenith-distance of the Sun). From the time that has elapsed after the rise of the planet called the Unnata, the shadow is to be computed as in the case of the Sun's shadow. Having thus computed the shadow or what is the same the zenith-distance of the Moon or that of the stars, the instrument called Nalaka could be pointed to the spot where that celestial body is situated.

Comm. The words Krānti, Spaṣtakrānti were explained before. Also the method of calculating the zenith-distance from a knowledge of the declination was described before in Trīpras'nādhikāra.

Verses 14, 15. Consideration of horizontal parallax in the case of visibility of the Moon or planets.

H cos z or what is called the Mahās'anku of the Moon or planet is to be reduced by $\frac{1}{15}$th of the respective daily motion gives the visible S'anku when the radius is taken to be 3438. If the radius is taken as 120, $\frac{1}{430}$th of the daily motion is to be subtracted. If H cos z is less than $\frac{1}{15}$th (or $\frac{1}{430}$th) of the daily motion, then the Moon is not visible. This applies to the other planets as well, but as this is a neglible matter, the earlier Āchāryas did not suggest this.

Comm. In the context of the subject of parallax, we mentioned that in the case of the Moon, the horizontal parallax happens to be $\frac{1}{15}$ of the Moon's daily motion

approximately. The same is extended to the case of the planets as well. To get the proportion in the case of taking the radius to be 120 only instead of 3438, rule of three is applied as follows. "If the radius be 3438, $\frac{v}{15}$ is parallax, what will it be if the radius be 120?" The answer is

$$\frac{v}{15} \times \frac{120}{3438} = \frac{v}{3438/8} = \frac{v}{430} \text{ very}$$

If H cos z be less than this $\frac{1}{15}$th or $\frac{1}{430}$th of the daily motion, the parallax does not allow the Moon to be seen. Bhāskara specifically talks about the Moon only because, the earlier Āchāryas did not apply the correction of parallax to the case of the planets, because it is not appreciable.

Verse 16. If an operation is neglected because its effect is not appreciable, or is not of much use, or because it is apparent, or if it implies great labour, or again if it implies a lot of exposition that would make the text unduly voluminous, ignoring the necessity of that operation should not be treated as wrong.

Comm. Here Bhāskara upholds the earlier Āchāryas not stipulating parallax in the case of planets, because it is not appreciable.

Here ends Graha-cchāyādhikāra.

GRAHŌDAYĀSTĀDHIKĀRA

Verse 1 *and first half of verse* 2.

The Udayalagna of a planet is termed Prāk-Dṛk-graha and the Aṣṭalagna is termed the Pas'chima-Dṛk-graha. If the Prāk-Dṛk-graha happens to be less than the current lagna the planet had already risen. If it be greater, the planet is still to rise. Similarly if the Pas'chima-Dṛk-graha is less than the current lagna, the planet had already set; otherwise is yet to set.

Comm. The words Prāk-Dṛk-graha and Pas'chima-Dṛk-graha are coined to indicate the points of intersection of the ecliptic with the eastern and western horizon respectively when the planet is rising or setting, when the planet has latitude. When the planet has no latitude Prāk-Dṛk-graha coincides with the rising planet and the Pas'chima-Dṛk-graha with the setting planet. When the planet has a latitude, the foot of the latitude, which signifies the planet's position on the ecliptic differs from the two Dṛk-grahas. In this case ie. when the planet has a latitude the Prāk-Dṛk-graha's longitude will be less than that of the planet's longitude whereas the Pas'chima-Dṛk-graha's longitude will be greater than that of the planet. If x and y be the differences of the longitudes of the planet and those of the Dṛk-grahas respectively, it is clear from a figure that $\tan x = \frac{\tan \beta}{\tan \theta}$ and $\tan y = \frac{\tan \beta}{\tan \phi}$ where β is the latitude of the planet and θ and ϕ the angles which the ecliptic makes with the horizon at the planet's rising and setting respectively.

Since θ and ϕ are then respectively the zenith-distances of the pole of the ecliptic in each case, and since

in the course of half a sidereal day, these zenith-distances could not be equal, x and y cannot be equal.

Second half of verse 2 *and verse* 3. To find the sidereal time in between the given time and the time of the planet's rising.

Find the time that has elapsed after the planet's rise, from a knowledge of the Iṣtalagna at a given time and Prāk-Dṛk-graha ie. the longitude of the rising point of the ecliptic at the time when the planet rises.

This time will be in Sāvana measure, because we have taken the planet's position at the given moment and not the position of the rising planet. From this time, knowing the daily motion of the planet of the day in question, obtain the arc; that would have been traversed in between the moments. Subtracting this arc from the planet's position at the moment, the planet's position at rising would be obtained approximately; approximately, because as per the procedure enunciated in verse eleven, Grahacchāyādhikāra, the Prāk-Dṛk-graha of the rising planet, and that obtained from the position of the planet at the moment differ. From this approximate time again compute the arc that would have been traversed by the planet during that time. Subtracting this arc which is nearer the truth from the planet's present position, we obtain a more approximate position of the rising planet. Again computing the Prāk-Dṛk-graha from this position, calculate the time from this Prāk-Dṛk-graha and the lagna of the moment. Repeating the process we will obtain the actual sidereal time in between the given time and the actual rising time of the planet. It is sidereal because, we have found the time as per the procedure of verse (11) referred to, where we have used तत्कालखेटोदय and not the actual ,खेटोदय, i.e the rising lagna of the planet obtained from the present position and not that of the rising planet.

This difference between the nature of the times was already dealt with.

The matter of this verse appears to have been unnecessarily complicated by Bhāskara but on careful scrutiny it is not so; for, the problem is to find the time that has elapsed after the rise of the planet upto a given time. Here the data are the given time and the corresponding position of the planet not on the horizon but elsewhere. We could not know immediately the time at which the planet rose. To find it alone, this method of successive approximation has to be used and there is no other go. Had we known the position of the planet while rising, the corresponding Prāk-Dṛk-graha could be found exactly and as this position of the Prāk-Dṛk-graha is a point of the ecliptic and the Iṣṭa-lagna ie. the rising point of the ecliptic at the given moment is also a point on the ecliptic, the time between the moments of rising of these two points of the ecliptic could be found in sidereal measure directly by noting the rising times of the Rāśis in between, without an appeal to the method of successive approximation.

Verse 4. The computation of the times of heliacal rising and setting of a planet in contradistinction to its rising time during a day on account of earth's diurnal rotation.

The times of rising and setting of a planet have been dealt with. Now I shall tell the procedure to be adopted in computing the times of heliacal rising and setting of a planet. If a planet has a daily motion less than that of the Sun, it rises heliacally in the east, and sets in the west; otherwise the reverse.

Comm. Take the example of the superior planets, say that of Jupiter. Since Jupiter moves slower than the Sun,

the Sun will have to overtake Jupiter and not vice versa. When it is the Sun that is to overtake, taking a position of Jupiter near the western horizon gradually the planet gets fainter and fainter as the Sun approaches him from west to east. Ultimately one evening the planet ceases to be perceptible within a particular distance of the Sun. This we call heliacal setting of the planet. Having thus set in the west, after a few days the planet rises in the east. This is so because, after the planet's heliacal setting, the Sun gradually approaches the planet from west to east; at a particular moment will have a longitude equal to that of the planet and then gradually gains in longitude over the planet. After a few days, in the eastern horizon, the planet will emerge from the rays of the Sun before Sunrise; it is not the planet moving west from east but it is the Sun moving from west to east so that the Sun recedes away from the planet towards east. The planet also will be having a motion from west to east but it being slower than that of the Sun, the latter gains over the former in its motion from west to east.

In the case of the inferior planets, say for example, Mercury, the velocity of Mercury is greater than that of the Sun; as such, Mercury will be overtaking the Sun and not vice versa. Thus in the eastern horizon, it is Mercury that enters the rays of the Sun, going from west to east, so that he sets in the east. After a few days of this heliacal setting, Mercury acquires a longitude equal to that of the Sun, and while overtaking the Sun from west to east, he emerges out of the rays of the Sun in the west, which is therefore heliacal rising. But, when Mercury is retrograde, the case is different. Some days after heliacal rising in the west, Mercury attains his maximum elongation and then begins to retrograde. The elongation then gradually decreases, and he will set again in the rays of the Sun in the west itself. A few days thereafter, still continuing retrograde, he emerges out of the Sun's rays

in the east, thus rising heliacally. After attaining the maximum elongation in his retrograde motion, his motion will then become direct, so that he again approaches the Sun, going from west to east. Next a little before overtaking the Sun, he sets heliacally in the east, and thereafter, having overtaking the Sun, he rises heliacally in the west.

Here, one point is to be mentioned. The time between Mercury's maximum elongation in the west and again the maximum elongation in the east (which are termed maximum eastern elongation and maximum western elongation with respect to the Sun) will be far less than the time between the maximum elongation in the east (ie. maximum western elongation with respect to the Sun) and that in the west (ie. maximum western elongation with respect to the Sun) in as much as when Mercury is retrograde, and the Sun having always direct motion, the relative velocity will be the sum of the retrograde velocity of Mercury and the direct velocity of the Sun.

In the case of the superior planets, as mentioned above, the superior planets set heliacally in the west and rise heliacally in the east. They will not rise heliacally in the west and will not set heliacally in the east which happens only if either the superior planet has a greater velocity than the Sun or the Sun has a retrograde motion which is never the case.

Verse 5. Speciality with respect to Mercury and Venus.

Mercury and Venus rise heliacally in the west in their direct motion, (attain maximum elongation before they) become retrograde, set heliacally there itself, then rise heliacally in the east continuing to be retrograde, (attain maximum elongation there before they) next become direct

and gradually set there (to rise again in the west) as before.

Comm. Explained above.

Verse 6. Kālāms'as or distance in degrees from the Sun within which the planets rise or set heliacally.

The Kālāms'as with respect to the Moon, Mars, Mercury, Jupiter, Venus and Saturn, or the degrees of distance from the Sun within which they rise or set heliacally are 12, 17, 14, 11, 10, 15 respectively. In the case of Mercury and Venus when they are retrograde the Kālāms'as are 12, and 8 respectively.

Comm. The Kālāms'as given above depend upon the luminosity of the respective planets. When Mercury and Venus happen to be retrograde, the Kālāms'as happen to be 2° less in each case because they are then nearest to the earth and as such being most luminous as seen by us will not set heliacally till they are very near the Sun.

Verse 7. To compute the moment when a planet rises or sets heliacally.

If it is to be known when a planet rises or sets heliacally the position of the Prāk-Dṛk-graha or the Pas'chima-Dṛk-graha as the case may be (Prāk-Dṛk-graha in case the rising or setting takes place in the east or the other in the other case) and that of the Sun also are to be computed on a day a little before the day of rising or setting as prognosticated by the S'īghra anomaly. In case the planet rises or sets in the west, to obtain the lagna the position of the Sun is to be increased by 180°.

Comm. Clear.

First half of verse 8. To obtain what are called Iṣṭa-Kālāmśas.

The time between the rising of the planet or of the Dṛk-graha and that of the Sun measured in ghatis multiplied by six gives what are called Iṣṭa-Kālāmśas.

Comm. Having found the approximate position of the Dṛk-graha as mentioned above when the planet is likely to rise or set heliacally, let the time in between the rising of this Dṛk-graha and that of the Sun (if it be the case of setting or rising in the west the position of the Sun is to be increased by 180° because the astalagna directed to be found in verse (1) is the point of intersection of the ecliptic with the *eastern horizon*, which is removed 180° from the setting point of the Sun) be multiplied by six. Since both the Dṛk graha and the Sun's position are points on the Ecliptic and since we have considered the time in between their rising moments, which is measured on the equator, and again since this time is measured in ghatis, the number of ghatis multiplied by six give the degrees, for, each ghati corresponds to six degrees of the equator (sixty ghatis corresponding to one sidereal day). These degrees are said to be Iṣṭa-Kālāmśas, which means that it gives the arc in between the feet of the declination circles of the Dṛk graha and the Sun at the Iṣṭa-kala ie. that particular time considered.

Second half of verse 8 *and verses* (9) *and* (10).

If the number of degrees so found ie. the Iṣṭa-Kālāmśas fall short of or exceed the number of Kālāmśas postulated for the rising or setting of the planet, then the planet's rising has to take place or has already taken place respectively and vice versa in the case of setting. The number of minutes of the difference of the prescribed and Iṣṭa-Kālāmśas multiplied by 1800 and divided by the rising time of the Rāśi expressed in Kalās and again

divided by the difference of the daily motions of the planet and the Sun expressed in minutes of arc if the planet is direct in motion or divided by the sum of those daily motions if the planet be retrograde gives the days elapsed after rising or to elapse for the rising to take place. Again, compute the positions of the Dṛk-graha and the Sun for the moment thus obtained and repeat the process till the actual moment is obtained.

Comm. Suppose the prescribed Kālāmśas for rising be x and suppose the Iṣṭa-Kālāmśas are y such that $y<x$; then for the planet to rise, the arc of the equator in between the feet of the declination circles of the Sun and the Dṛk-graha has to increase for the planet to rise which means that this takes some more time to happen. Similarly if $y>x$, the planet has already risen. The question of the arc decreasing does not arise in this case of rising, because the planet's position in the east is behind that of the Sun, and in the case of a superior planet the Sun has to advance further for the planet to rise ie. the arc has to increase and in the case of a retrograde inferior planet also, the arc will be increasing. In the case of an inferior planet being direct, the arc will be decreasing no doubt, but we have to remember that this is a case of an inferior planet *setting* and not *rising* which we are considering. In the case of setting, in the east, however, of the inferior planet the moment of setting has already elapsed and thus the condition is reverse to that of rising. The case of setting of a superior planet in the east never happens. In the case of setting of a superior planet in the west, if the Iṣta-Kālāmśas be less than the prescribed Kālāmśas, ie. $y<x$, setting must have already taken place, the Sun having approached the planet from behind and already effected heliacal setting. Thus this is also reverse to the condition of rising as stated. In the case of an inferior planet setting in the west if $y>x$, the planet should have set already since the inferior planet is retrograde while setting in the west.

This condition is also reverse to what has been stated in the case of rising. The case of a superior planet rising in the west also never happens, because it is the Sun that overtakes the planet and also the superior planet cannot be retrograde while near the Sun.

The case of rising which has already elapsed the Iṣṭa-Kālāms'as y will be evidently greater than x ie. $y > x$. To obtain the time by which the rising will take place after the moment in question, ie. the rising of a superior planet which is direct and an inferior planet which is retrograde, we have to take the difference of the prescribed and Iṣṭa-Kālāms'as and find out the time by rule of three as follows.

(1) Since we have to find the corresponding arc of the ecliptic in minutes of arc, from $(y-x) \times 60'$ the difference of the Kālāms'as of the equator converted into minutes ie. asus we have first to use the proportion. "If by the rising time of the Rās'i t in asus in which the planet is situated we have $1800'$ of the ecliptic, what will we have by $(y-x) \times 60$? The answer is $\frac{(y-x) \times 60 \times 1800}{t}$.

Next we have to find the days when this arc of the ecliptic is covered by the proportion. "If the Sun overtakes the planet by $(u-v)$ minutes of the ecliptic per day, how many days are taken to cover the above arc?" The answer is as stated.

The answer gives in days together with fraction of a day when the planet is likely to rise. We have to repeat the process because the motions of the planet as well as the Sun differ from moment to moment and the time calculated above by rule of three taking into account their motions for the entire day will be only approximate. The computation in the case of setting of a planet may be similarly considered, the setting of a superior planet in the or that of a direct inferior planet in the east.

Verses 11 *and* 12. If the Prāk-Dṛk-graha has a longitude greater than that of the Sun or the Pas'chima-Dṛk-graha has one less than that of the Sun, the sum of the prescribed Kālāms'as and Iṣta-Kālāms'as converted into minutes of arc will have to be used to compute the elapsed days or the days after which the respective phenomenon is going to occur.

If the Iṣta-Kālāms'as y be greater than the prescribed namely x, the reverse to what has been stated in the second halt of verse (8) and in the first line of verse (9) happens ie. a phenomenon which has elapsed when $y<x$ will happen in the future and vice versa.

Comm. If the Prāk-Dṛk-graha has a longitude greater than that of the Sun, (since the prescribed Kālāms'as are between the planet and the Sun, the planet being behind the Sun, and the Iṣta-Kālāms'as are now on the other side of the Sun) the planet has advanced over its position of heliacal setting be it a superior planet or inferior by the sum of the prescribed Kālāms'as and Iṣta-Kālāms'as. Hence the days that have elapsed after the heliacal setting have to be calculated with the sum of the two Kālāms'as. If it be also a case of a retrograde inferior Prāk-Dṛk-graha rising, having a longitude greater than that of the Sun, even then the prescribed Kālāms'as are behind the Sun and the Iṣta-Kālāms'as ahead of the Sun. So from the position of the planet's heliacal rising, the Sun and the planet have advanced in opposite direction to a distance which is the sum of the prescribed Kālāms'as and Iṣta Kālāms'as. Hence computation has to proceed with the sum of the two Kālāms'as to get the time that has elapsed after the planet's heliacal rising. Similar is the argument for the Pas'chima-Dṛk-graha.

We have commented on verses from the latter half of (8) upto (10) where the Iṣta-Kālāms'as y happen to be less

than x. If $y>x$, or again the Prāk-Dṛk-graha has a longitude greater that of the Sun, the phenomenon of rising or setting which happened when $y<x$, will have to be taken as going to happen and that which was going to happen when $y<x$, must have happened already. It is enough to consider just one case instead of all the four cases since the argument is similar as before. Let us take $y>x$ and it is an inferior planet in the east behind the Sun. According to the latter half verse (8) the rising had to take place y being less than x, whereas, now, y being greater than x, rising had already taken place, in contradistinction to what happens when $y<x$.

Here ends the Udayāstādhikāra.

SṚNGONNATYADHIKĀRA

Verse 1. Either in the last quarter of a lunation, or in the first quarter, on the day when the elevation of the cusps of the crescent Moon is to be determined, then either at the moment of Moon-rise or Moon-set or (for the matter of that during any part of the night) the Hsine of the altitude of the Moon is to be computed by noting the time from the moment of Moon-rise.

Comm. Either in the first quarter or the last quarter of the lunation, ie. when the phase of the Moon according to the definition of phase in modern terms is less than half, the Moon will be a crescent. Also, generally, on the back-ground of the horizon, we notice that one of the cusps is more elevated than the other. This elevation goes by the name Sṛngonnati.

Even in the middle half of the lunation, Bhāskara mentions that Brahmagupta and some others (meaning Sripati whom he closely follows) attempted at finding the elevation (strictly speaking elevation during the second quarter and depression during the third quarter) of the dark horns. Bhāskara does not appreciate this since, nobody would think about this as it does not appeal to the eye at all.

Verse 2. To find the Hsine of the altitude of the Sun.

The Hsine of the altitude of the Sun is to be com-puted, assuming the rising Sun to be in the opposite hemisphere, south or north (ie. if he be originally in the northern, assume him to be in the south) and using the formula given in verse 54 of Tripras'nādhyāya "अथोन्नता-दूनयुताञ्चरेण" given the time measured in asus that has elapsed after Sunset.

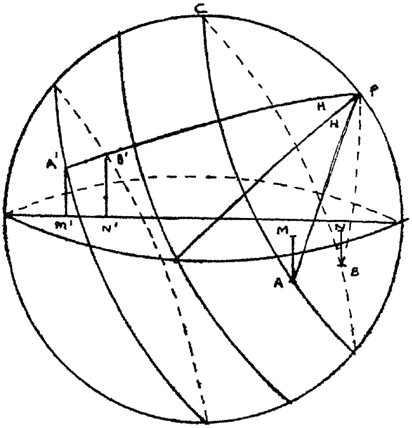


Fig. 112

Comm. During the early part of the night or during the latter part thereof, when the Sun is below the horizon, the Sun will be occupying symmetrical positions with respect to the horizon at times which are equally removed from Sunset and the next Sun-rise. This is clear from the figure 112. Let A and B be two such symmetrical positions where AM and BN are the Hsines of the altitudes.

Evidently AM = BN, $C\hat{P}A = C\hat{P}B$. Since the rising eastern hour-angle of the Sun equals he setting western hour-angle, and since $C\hat{P}A = C\hat{P}B$, the time elapsed after Sunset when the Sun is at B, will be equal to the time before Sun-rise in the position A. Hence by congruence -AM = BN = Hsine of the altitudes in the two symmetric positions. Using the modern formula from the triangle PZS, $\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos h$, when

$C\hat{P}A = C\hat{P}B = h$ cos z will be the same in the two positions A and B. Putting $z = \overline{90 + \theta}$, $\cos z = -\sin \theta$ will be the same ie. H $\sin \theta$ will be the same in the two positions ie. the S'ankus will be the same. In the formula

$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos h$

Put $z = 90 + \theta$, $h = 90 + H$, $\delta = \delta$ in the positions A, B and $z = 90 - \theta$, $h = 90 - H$. $\delta = -\delta$ in the positions A′, B′

We have then $-\sin \theta = \sin \phi \sin \delta - \cos \phi \cos \delta \sin H$

and $\sin \theta = -\sin \phi \sin \delta + \cos \phi \cos \delta \sin H$

which are identical. This means that the altitude θ below the horizon with $+\delta$ in the positions A, B is computable with $-\delta$ in the positions A′, B′. This accounts for the statement made 'गोलविपर्यये'

The second statement of the latter half of verse (2) says S'ankutala $=$ S'anku $\times \frac{s}{12}$ which we have proved in Tripras'nādhyāya.

Verse (3) *and first half of* (4). To obtain the Bhuja of the Sun.

The S'ankutala of the inverse altitude or the altitude below the horizon, is north (in contradistinction to what it is above the horizon). The sum or difference of the Agrā and S'ankutala according as they are of the same direction or of opposite directions, is the Bhuja. The sum or difference of the Bhujas of the Sun and Moon, according as they are of opposite or the same directions is what is called the Spaṣṭa Bhuja whose direction is to be construed as that of the Moon. If the Bhuja of the Moon falls short of that of the Sun, then the direction of the Spaṣṭa Bhuja is that opposite to that of the Moon.

Comm. In fig. 113, we have shown five different positions of the Sun S_1 to S_5 the feet of the S'ankus being B_1 to B. From these feet of the S'ankus draw perpendiculars on the Udayāstasutras as well as on the East-west line their points of intersection being respectively A_1 to A_5 and C_1 to C_5. The perpendiculars from the feet of the S'ankus on the East-west line go by the name Bhujas, the perpendi-

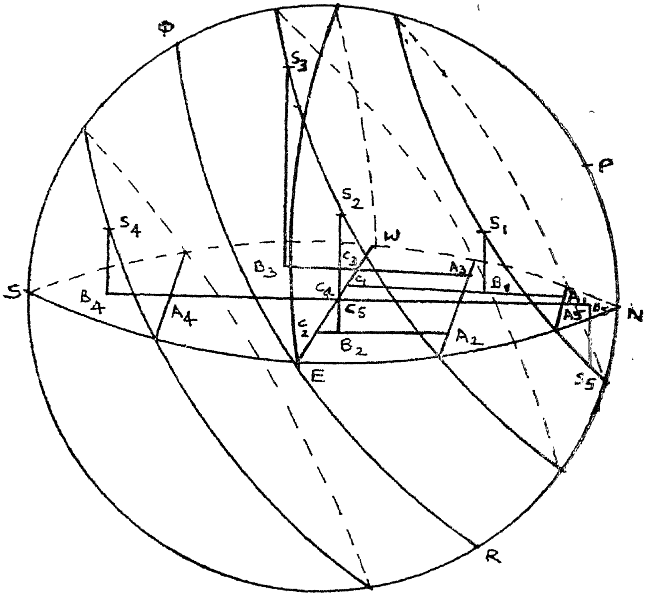
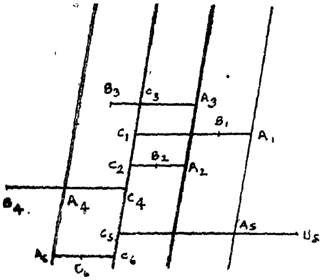


Fig. 113

cular distances between the Udayāstasutras and the East-west line are called Agrās and the perpendiculars from the feet of the S'ankus on the Udayāstasutras are called S'ankutalas which were all defined in the course of the Tripras'nādhyāya. We have from the spherical triangle PZS, $\sin \delta = \sin \phi \cos z + \cos \phi \sin z \sin a$ so that $\frac{\sin \delta}{\cos \phi} = \tan \phi \cos z + \sin z \sin a$ which was shown as $A = S + B$ in the Tripras'nādhyāya ie. Agrā = S'ankutala + Bhuja. Changing δ into $-\delta$ and a into $-a$ we have different formulae, the standard form being $A = S + B$. We don't propose to change ϕ into $-\phi$, because in India this case does not arise. Thus our latitude being north, the S'ankutalas defined above as the distances from the feet of the S'ankus from the Udayāstasutras are always deemed as south. It need not be reiterated that the Udayāstasutras

are the straight lines joining the rising and setting points of the celestial body. As such, these Udayāstasutras are all parallel to the East-west line. In fig. 114, in the horizontal plane containing the points A's, B's and C's in the respective cases, according to the Hindu convention, we talk of Agrā and Bhuja as being Uttara Agrā, Dakṣināgrā, Uttara Bhuja and Dakṣina Bhuja. S'ankutalam is always taken to be south except in the fifth case shown in figs. 113 and 114 when the altitude happens to be below the horizon. In this case the S'ankutala is spoken as north. Thus the general formula A=S+B assumes the following various forms, in the respective cases.



Fig, 114

(1) Uttarāgrā—Dakṣina S'ankutala = Uttara Bhuja. This corresponds to A, B, C of fig. 114 where AC, is Uttarāgrā, BA, Dakshina S'ankutala and BC, Uttara Bhuja.

(2) In the case of $A_3 B_3 C_3$, S'ankutala—Agrā = Dakṣina Bhuja since $B_3 A_3 - C_3 A_3 = B_3 C_3$. This occurs after the Sun's diurnal path has crossed the prime-vertical and the Sun has a position on the south of the prime-vertical, having a northern declination. In the case of $A_4 B_4 C_4$, $B_4 C_4 = B_4 A_4 + A_4 C_4$ ie. Dakṣina Bhuja = Dakṣina S'ankutala + Dakṣināgrā. In the fifth case when the Sun's altitude is below the horizon, $B_5 C_5 = B_5 A_5 + C_5 A_5$ ie. Uttara Bhuja = Uttara S'ankutala + Uttarāgrā. In the sixth case when the Sun has a southern declination and an altitude below the horizon, $B_6 C_6 = A_6 C_6 + A_6 B_6$ ie. Dakṣina Bhuja = Dakṣināgrā minus Uttara S'ankutala. In the case of $A_2 B_2 C_2$, Uttarāgrā =

Dakṣina S'ankutala + Uttara Bhuja, since $A_2 C_2 = B_2 A_2 + B_2 C_2$. In the absence of a unifying convention, all these formulae are loose and apt to create confusion. So we shall unify all these formulae into the standard form A=S+B which holds good universally, if we have the conventions.

(1) Agrā shall be deemed positive if it be north and negative if it be south;

(2) Similarly with respect to the Bhuja. But, with respect to the S'ankutala, we shall deem it positive if it be south and negative if north. These conventions comprehend all the cases and unify them into the standard form A=S+B. The corresponding conventions with respect to δ, a are that $+\delta$ corresponds to Uttarāgrā and $+a$ corresponds to Uttara Bhuja.

Having the above conventions, and finding the Bhujas of the Moon above the horizon and the Sun below the horizon, the Spaṣta Bhuja required in finding the S'ṛngonnati in the present chapter, is defined as the difference of the Bhujas of the Sun and the Moon and the sum thereof according as they are of the same direction north or south or of different directions. Also this Spaṣta Bhuja is by convention said to have the same direction as that of the Moon. If, however, the Bhuja of the Moon falls short of that of the Sun, then the Spaṣta Bhuja is said to have the direction opposite to that of the Moon.

Verse (4). Definition of Koti.

I deem that the Koti should be taken as the sum of the S'ankus of the Sun and the Moon, the one being below the horizon and the other above respectively.

Comm. We have defined above the Spaṣta Bhuja as the north-south distance between the feet of the S'ankus

of the Sun and the Moon. Herein Bhāskara defines the Koti as the sum of the S'ankus of the Sun and the Moon which is the vertical distance between the points on the armillary sphere which represent the Sun and the Moon.

Bhāskara says 'I deem' to signify that he differs from Brahmagupta in this, who defines the chord joining the Sun and Moon on the armillary sphere which is equal to 2 Hsine of the SM on the sphere as the Karṇa. Brahmagupta defines the Bhuja and Karṇa from which he deduces the Koti as $\sqrt{Karṇa^2 - Bhuja^2}$. Bhāskara argues that since the Karṇa defined by Brahmagupta is not in the vertical plane, since the Sun and the Moon are not in the same vertical plane, so, the Koti defined by him throught he Karṇa will not be in the vertical plane. The Karṇa defined by Bhāskara is not on the other hand the chord joining the Sun and the Moon but lies in the vertical plane in which the perpendicular from the Moon's position on the armillary sphere, on the horizontal plane eontaining the Sun's position on that sphere. Also the Bhuja defined both by Brahmagupta and Bhāskara is the projection on the north-south line of the join of the Sun's position on the sphere and the foot of the perpendicular from the Moon's position on the horizontal plane through the Sun's position and it is not actually the above join. Bhāskara is evidently guided by the right angled triangle SMN which is not strictly a spherical triangle as per its modern definition as the arc SN is not that of a great circle. This figure guided him to take the Bhuja horizontal and the Koti vertical so that his Karṇa is also in a vertical plane. Bhāskara's

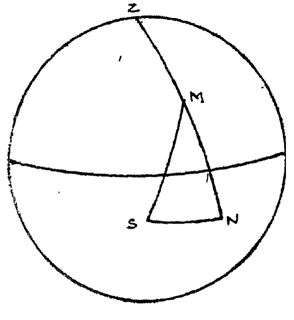


Fig. 115

Karṇa therefore has nothing to do with the arc SM but only has the virtue of being in a vertical plane. Sripati adopted Brahmagupta's method. Kamalākara neither follows Brahmagupta nor Bhāskara but follows a method of his own. In fact the methods adopted by these ancient Hindu astronomers are not mathematically correct because the question of determining the cusps as well as phase of the Moon is concerned with the actual positions of the Sun, Moon and the earth in space and not as seen on the sphere. Hence M. M. Sudhakara Dwivedi has written a small book by the name Vāstava-Sṛngonnati following modern methods. The modern method which gives the truth of the matter is depicted in standar modern texts.

Verse 5. The hypotenuse or Karṇa is the square-root of the sum of the squares of the Bhuja and Koti. The Bhuja multiplied by 6 and divided by the Karṇa gives what is known as the Dik-valana of the Moon. The direction of the Valana has the same direction as the Spaṣṭa Bhuja defined before.

Comm. The idea is that taking the radius of the Moon's disc to be six angulas or units, representing the Karṇa, the magnitude of the Bhuja on the same scale gives a measure of what is defined as Valana. Thus Valana $= \frac{6B}{K}$ where B and K stand for the Bhuja and Karṇa defined before. The idea of this Valana will be clarified in the ensuing verses.

Verse 6. The Hsine of the elongation of the Moon is to be multiplied by the radius vector of the Moon measured in Yojanas, and divided by the radius vector of the Sun, also measured in Yojanas; the arc of the Hsine so obtained is to be added to the longitude of the Moon in the bright half of the lunation and is to be subtracted from the same in the dark half.

Comm. This is a correction to be made in the longitude of the Moon to depict graphically the phase of the Moon. Bhāskara says that many of the prior astronomers took that the phase of the Moon was in direct proportion to the elongation of the Moon. Taking the radius of the Moon's disc to be six angulas or units, and assuming that when the elongation is 180°, the entire disc being illuminated, it was thought that 12 units of the diameter correspond to 180° of elongation so that the S'ukla of the disc measured by the central width of the illuminated disc increases at the rate of 1 unit for 15° of elongation. (The word S'ukla may be taken to correspond to the modern word phase. S'ukla is expressed in angulas, taking the diameter of the disc to be 12 angulas. The measure of the portion of the diameter of the Moon's disc perpendicular to the diameter which is the join of the extremities of the cusps, covered by the illuminated part of the disc, (which may be defined as the maximum width of the illuminated part of the disc) measured in angulas is said to be the S'ukla. The word 'phase' is used to signify the ratio of the above width to the diameter. S'ukla is expressed in angulas, whereas phase is expressed as a ratio. The S'ukla is equal to twelve times phase). Bhāskara rightly argues that this method of measuring the S'ukla is approximate because he says that six angulas of S'ukla is had when the elongation is not 90°, but only 85°-45′. This may be substantiated as follows from fig. 116. Let E, M, S stand for the earth, Moon and the Sun. Let $\widehat{EMS}=90^\circ$ so that it is a moment of dichotomy ie. the moment when the phase is half and the S'ukla 6 angulas. Let θ be then the elongation of the Moon, so that $\cos\theta = \frac{EM}{ES}$. Taking the average values given for EM and ES by Bhāskara,

$$: \frac{51566}{689377} \quad \frac{19}{254} \text{ (obtaining a convergent)} = .0748.$$

From tables, we find $\theta = 85°–43'$ which Bhāskara takes to be $85°–45'$.

Fig. 116

In the wake of this, Bhāskara tries to make amends in the approximate formula prescribed to obtain the S'ukla. He prescribes addition of $4°–15'$ to the longitude of the Moon in the bright half, and subtraction in the dark. In between the moment of conjunction and the moment of dichotomy, he derives the following formula (vide fig. 117).

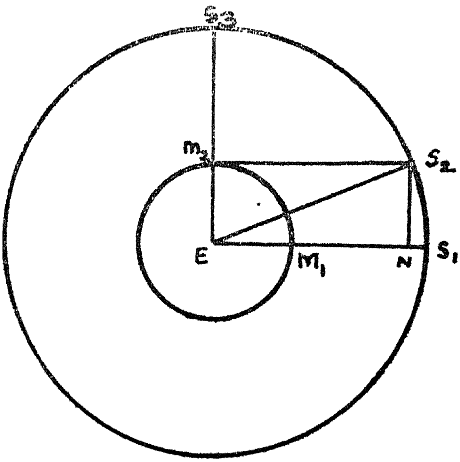


Fig. 117

The deficiency of $4°–15'$ is had in the form of the angle $\}_1$. When the Sun is at S_1 the Moon being at M_1, it

is the moment of conjunction. When the Sun has moved from the point S_1 to S_3, $S_1 \hat{E} S_3$ being 90°, there is a deficiency of magnitude $S_1 E S_2$. In other words, when $S_2 N$ has assumed the position $S_3 E$ there is a deficiency of 4°–15′ ie. for an increase of 90° of elongation, there is a deficiency of 4°–15′ in the longitude of the Moon. Hence, for the Hsine to become the radius, there corresponds a portion $E M_2$ or $S_2 N$, which is the Hsine of 4°–15′, so that the following rule of three is adopted. 'If the Sun's distance $E S_3$ corresponds to the radius, what does $E M_2$ the distance of the Moon correspond to?" The result is $\frac{m}{s} \times R$, m and s being the respective distances. Then the following proportion is used "If by H sin ξ equal to R, ξ being the Moon's elongation, we have mR/s, what shall we have for an arbitrary H sin ξ?" Thus the answer is

$$\text{H} \sin \xi \times \frac{mRs}{R} = \text{H} \sin \xi \frac{m}{s}. \quad \text{H} \sin^{-1} \left(\frac{m}{s} \times \text{H} \sin \xi\right)$$

where m and s are the distances of the Moon and the Sun, is to be added to the longitude of the Moon or to be subtracted as the case may be, to have the rectified longitude of ths Moon from which the phase is to be calculated according to Bhāskara.

Here we are to offer the following remarks. No Bhāskara was correct in estimating the moment of dichotomy to be that when ξ the Moon's elongation is not 90° but 85°–45. But the amended formula is not the correct mathematical form. The modern formula to find the phase is $\frac{1+\cos \text{EMS}}{2}$ aud since from fig. 116, SM is nearly equal to SE, so EMS is very nearly equal to

180–MES, so that $\frac{1+\cos EMS}{2} = \frac{1-\cos MES}{2}$ which may be taken to be an approximate truth. This formula was indeed given by Lallācārya in the following verse, long before Bhāskara "रविशीतकरान्तरांशजीवा विपरीता शशिखण्डताडिता च, विहृता त्रिभजीवया सितं स्यात् शशलक्ष्माङ्गवदङ्गुलानि तस्मिन्" verse 12 (Candra S'ṛngonnatyadhikāra). Here विपरीता रविशीतकरान्तरांशजीवा mean Hvers ξ = R—H cos ξ. शशिखण्डताडिता = multiplied by the radius of the disc of the Moon. विहृता त्रिभजीवया = divided by R. Thus the formula given by Lallācārya amounts to $r\,\frac{(R-H\cos\xi)}{R} = \frac{d}{2}(1-\cos\xi)$ where r gives the angulas in the radius of the Moon's disc and d the diameter, and ξ = elongation of the Moon. Since the definition of phase in modern terms is a ratio, namely the ratio of the maximum width of the crescent to the diameter, phase $\times\, d$ = S'ukla of the Hindu astronomers

$$\therefore\quad \frac{1-\cos\,(\text{elongation of the Moon})}{2} \times d = \text{Sukla}$$

$= r\,(1-\cos MES)$ as given by Lallācārya. So Lallācārya's formula is quite correct. We shall trace Lallācārya's steps in obtaining such an intricate correct formula which was overlooked by Bhāskara. (Refer fig. 118) Let E be the earth, S the Sun and M_1, M_2, M_3 etc. the positions of the Moon as the elongation gradually increases. No doubt the S'ukla increases with elongation as known to all Hindu astronomers. But, the question is, does it increase with the sine or versine? We know both the sine and versine increase with the angle. Lallācārya noticed that Sm_1, Sm_2, Sm_3 as the Moon occupied positions M_1, M_2, M_3 etc., are the Hindu versines which are increasing. So he postulated that the S'ukla increases with the Hindu versine. His formulation was thus correct. But why Bhāskara overlooked this correct formula was traceable to his getting prejudiced

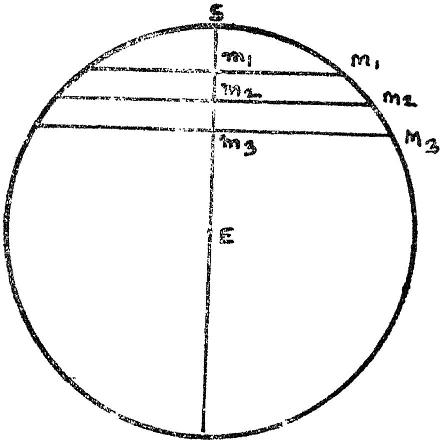


Fig. 118

against Lallācārya's some other formulae which used the Hversine where he ought to have used the Hsine. For example in the case of the Valana Lallācārya's mistake is traceable to his confusion as to whether he was to choose the Hsine or Hversine when both increase as the angle increases.

Verse 7. Graphical depiction of the cusps.

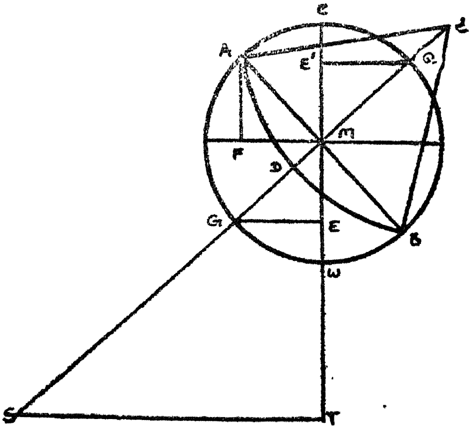


Fig. 119

Let in fig. 119, ST be the Bhuja, MT the Koti and SM the Karṇa formerly defined. As per verse (5) B = Valana where B = ST, K = SM, and 6 = MG so that $GE = \frac{6B}{K}$ = Valana, which is in practice drawn as E′G′ from the east point *e*, in the form of a Rsine. CG goes by the name Valanasūtra. Compute the S'ukla in angulas after making the prescribed correction in the longitude of the Moon as stated by Bhāskara and dividing the elongation of the Moon, thereafter obtained, by 15. Mark off the S'ukla in angulas along the Valanasūtra from G. Suppose GD is the S'ukla. Draw the diameter AB perpendicular to the Valanasūtra passing through M. Draw the circle circumscribing D, A, B. Its centre lies evidently on the Valanasūtra, say C. The circle drawn is called Parilekha Vṛtta, its radius CA is called Parilekhasūtra and the point C Parilekha Vṛtta Madhya. In the triangle CAM, which is right-angled CA is the Karṇa, AM the Bhuja and MC the Koti. CD is equal to the Karṇa CA so that MD = CD − CM = Karṇa − Koti = $K - k$ (say). We have $AM^2 = CA^2 - CM^2 = K^2 - k^2 = 36 = B^2$; Hence $K + k = \frac{B^2}{K-k}$.

Here $K-k$ = MD is known because GD the S'ukla is known. So $\frac{B^2}{K-k} = K+k$ is known. Thus knowing $K-k$ and $K+k$, by using what is called Samkramagaṇita ie. by adding $K+k$ and $K-k$, we have K and by subtracting we have k. Here the Koti CM is called Vibhā and the Karṇa CD the Swabhā. The Parilekhasūtra or the radius of the Parilekha Vṛtta being the Karṇa, the Swabhā is thus the Parilekhasūtra.

In the wake of this exposition, the translation of verse (7) runs as follows. "Let the compliment of the elongation (corrected as directed in verse (6)) divided by 15 be the denominator; let the numerator be 36; take the

result after division and put it in two places; with this and the complement of the elongation divided by 15, using Samkramagaṇita, we have successively Vibhā and Swabhā".

Comm. Here, the meaning of taking the complement of the elongation and dividing by 15 is to obtain the magnitude of MD directly; for, $\frac{90-e}{15} = 6 - e/15 = MG - e/15 = MG - GD = MD$ where e is the elongation and $e/15$ is the S'ukla. In other words, finding the S'ukla by dividing the elongation by 15 and subtracting it from the radius to get MD is the same as taking the complement of the elongation and dividing it by 15 to obtain MD.

Bhāskara quotes in the course of the commentary, the verse from his Leelāvatī, namely "भुजाद्वर्गितात् कोटिकर्णान्तरान्तम् द्विधा कोटिकर्णान्तरेणोनयुक्तम्। तदर्धे क्रमात्कोटिकर्णौ भवेताम्, इदं धीमताऽऽवेद्य सर्वत्र योज्यम्" which means $\frac{B^2}{K-k} = K+k$ since $K^2 - k^2 = B^2$. This situation arises when the Bhuja B and the difference of Koti and Karṇa ie. $K - k$ are given and it is required to find K and k separately.

Verses (8) and (9). Draw a circle with radius 6 angulas to represent the disc of the Moon. Mark the disc with the Cardinal points signifying the east, west, north and south. Compute the Valana as directed in verse (5) (which represents E′ G′ in fig. 119) and mark it off as a Hsine from the west point w in the last quarter and from the east point e in the first quarter. From the centre M of the Moon's disc, mark off Vibhā MC (computed as directed) along the join of MG′. With centre C and radius Swabhā (already computed as directed) draw a circle. The spherical sector of the Moon's globe thus demarked by the arc of the circle ADB namely AGBD is found to have the elevated cusp in the opposite direction in which the Valana has been marked.

Comm. The directions are clear in the wake of the previous commentary. In marking off the Valana, Bhāskara says 'from the west in the last quarter and from the east in the first quarter'. In the first quarter, the Moon will be visible immediately after Sunset in the western sky and the illuminated part of the Moon will be towards the western side of the disc in which direction the Sun is situated. The eastern point and the western point of the disc are easily discernible on the background of the horizon. Drawing first the diameter 'WE' of the disc along the vertical circle in case the Moon is due west or else along a small circle parallel to the prime-vertical called Upa-Vṛtta, we have the east point of the disc namely *e* vertically above the disc in the first quarter. (The figure 119 is shown for the first quarter). Then we are directed to mark off the Valana E′ G′ in the form of a Hsine'. Since the Valana is already computed as directed in verse (5), to mark off this Valana we have to lay the scale perpendicular to WE. Such a point G′ on the circumference of the disc is to be marked from which the Hsine equals the Valana computed. It is to be noted here that it is not marking G′ E′ from G′, for, G′ is not known, nor E′ for the matter of that, but locating G′ knowing the magnitude of the Valana. Though we happened to mention before that GE is the Valana, in the light of what Bhāskara mentions in verse (8) we have to understand that in the first quarter, the practice is to mark off Valana from the point *e* marked on the disc before hand. Though it does not matter when we take EG to be the Valana E′ G′, we have to notice the practice followed. The direction given in the course of the commentary "केन्द्राद्वलनोपरिवृत्तात बहिरपि खटिकया सूत्रमुच्छ्राद्यम्" meaning thereby that from the point M, we have to mark off along MG′ where G′ is the point pertaining to Valana, a line with the help of a chalk and a fine thread, namely MC. This practice is still in vogue adopted by masons to draw straight lines.

It is interesting to note Bhāskara waxing poetic in the course of the commentary under these verses describing how the illumination of the Moon's disc by the rays of the Sun is effected. The description is quite apt, interesting and scientific reminding us of Varāhamihirācārya's description in the verse 'सलिलमये शशिनि etc.' in Bṛhat-Saṁhitā. Of course, following Varāhamihira Bhāskara also takes the Moon's globe to contain water within its bosom which Modern Science has yet to confirm.

Verses 10, 11, 12. The Koti and Karṇa defined by Brahmagupta, do not lead us to accordance between computation and observation to locate the cusps. I request good mathematicians to verify this carefully. In a place where the latitude is $(90-\omega)=(90-24)=66°$, when the ecliptic coincides with the horizon, and when the Sun is in the beginning of Mēṣa which is then rising in the east, and the Moon in the beginning of Makara, then the Moon is dichotomized by the meridian and the illuminated part of the Moon's disc is towards the east. This does not hold good according to Brahmagupta's definition of Koti. because then the Bhuja as well as Koti according to his definition is equal to R. When the Bhuja is zero, the cusps will be horizontal, and when the Koti is zero, they will be vertical. Brahmagupta's Bhuja and Koti both being equal to R, the cusps therefore cannot be vertical which is against truth as stated above. Why should I make this statement? May my homage be to those great.

Comm. When the latitude of a place be $90-\omega$, and when the ecliptic coincides with the horizon, it is clear that in whatever positions be the Sun and the Moon (assuming that the Moon is also approximately on the horizon) the cusps of the Moon are always vertical, there being Bhuja only and no Koti. But according to Brahma-

gupta's definition of Karṇa, it will be in this particular case. $\sqrt{R^2 + R^2} = R\sqrt{2}$ so that the Koti will be $\sqrt{2R^2 - R^2} = \sqrt{R^2} = R$, where the Bhuja also is R (projection of the line joining the Sun and the Moon on the north-south line. Though in the verse (11) above one particular position of the Sun and one of the Moon, are contemplated, the same argument holds good, says Bhāskara in the course of the commentary, whatever positions are occupied by the Sun and the Moon on the ecliptic. In this case there is only Bhuja existing and no Koti, so that the cusps will be vertical. But in all these cases, there is Koti according to Brahmagupta's definition, so that the cusps will not be vertical according to him, which is clearly against truth.

Here we have to note that in the example cited by Bhāskara the Bhuja equal to R is along the north-south direction, and even though the Koti according to Bhāskarajs definition is conceived to be vertical, the Karṇa according to Brahmagupta's conception being ES where E and S are the east and south points, his Koti will be horizontal coinciding with OE, O being the centre of the horizontal ecliptic. Further according to Brahmagupta, the join of the cusps will be perpendicular to the Karṇa which does not therefore go against truth. Similarly in all the cases wherever be the Sun and the Moon on the horizon, Brahmagupta's Karṇa being a line joining the centres of the discs of the Sun and the Moon, it will be a horizontal line and so the cusps could be vertical. It is not clear whether or not Bhāskara recognized this namely that the Karṇa and Koti of Brahmagupta in the cases cited above are all horizontal lines so that the cusps could be vertical even according to Brahmagupta. That is why he pays homage to Brahmagupta in the last line of (12). Probably Bhāskara expected that Brahmagupta's Koti also should have been vertical as his. The fact that he recognized that Brahmagupta's Koti will not be vertical

but inclined, is justified here since the Kotis in all the cases cited are all horizontal lines.

In fact, as we stated before, even Bhāskara's analysis is not sound, for which reason, M. M. Sudhākara Dwivedi, wrote the booklet called Vāstava Sṛngonnati based on modern lines. As this topic could be found in books of modern astronomy, we need not go into the treatment of Sudhākara Dwivedi here.

Here ends the Sṛngonnatyadhikāra

GRAHAYUTYADHIKĀRA

Verse 1. The mean diameters of Mars, Mercury, Jupiter, Venus and Saturn are respectively 4′–45″, 6′–15″, ′–20″ 9′-5

Comm. Bhāskara gives us a method under verse (5) of Chandragrahaṇādhikāra, as to how the angular diameters of celestial bodies were being measured with an instrument that we may call 'protractor', describing it as follows. "यस्मिन् दिने अर्कस्य मध्यतुल्या स्फुटा गतिः स्यात् तस्मिन् दिने उदयकाले चक्रकलाव्यासार्धमितेन यष्टिद्वितयेन मूलमिलितेन तत्रस्थदृष्ट्या तदग्राभ्यां बिम्बप्रान्तौ विध्येत् "या यष्ट्यग्रयोरन्तरकलाः ता रविबिम्बकला भवन्ति मध्यमाः" ie. On the day on which the true motion of the Sun equals the mean, *in the morning*, observe with an instrument having two equal rods jointed at one end (and the other ends being connected by a flexible protractor marked with minutes and seconds of arc) placing the eye at the joint of the rods, and the two rods pointing to the extremities of a diameter of the disc. The magnitude of the disc is then read on the protractor, which gives the mean diameter".

This method is alright so far as it goes. The measuring being done at the time of morning and that too with the naked eye might have been responsible for the exaggerated magnitudes of the diameters of the discs of the planets, as compared with their modern values. This kind of exaggerate estimate is due what is called the phenomenon of irradiation which increases the apparent size of a brilliant body when seen at some distance. Even Tycho Brahe, an accurate and brilliant astronomer prior to the invention of the telescope gave estimates of these angular diameters which are nearly the same as remarked by Burgess in his translation of Sūryasiddhānta under verses 13, 14 ch. VII.

Verse 2. These estimates being multiplied by the difference of the radius and Sīghrakarṇa and divided by thrice the Sīghra-antyaphlajyā are to be added or subtracted from the mean values above given according as the Sīghrakarṇa is less or greater than the radius to give the rectified values. Three minutes of arc are to be construed as one angula in this respect.

Comm. When the Sīghrakarṇa equals the radius we know that the planet is situated at the mean distance. The word planet here stands, of course, for the Mandaspaṣṭagraha which may be roughly taken to be the mean planet, the equation of centre being small. If the Sīghrakarṇa falls short of the radius, then evidently the planet is nearer the earth than the mean position so that disc of the planet appears to be bigger. Otherwise, the planet is further and its disc appears to be smaller. It was noted that approximately there was an increase or decrease $\frac{1}{3}$ of the magnitude of the disc by the decrease or increase of the Sīghrakarṇa by the antyaphalajyā. Hence, in between the two positions, rule of three is used to obtain the magnitudes as follows. "If by a difference of the Sīghrakarṇa and radius equal to the antyaphalajyā, there is a difference of $\frac{1}{3}$ of the actual magnitude, what would it be for an arbitrary difference ?" The answer is $d \times \frac{1}{3} \times \frac{1}{a} = \frac{d}{3a}$ where d = Sīghrakarṇa or radius and a the antyaphalajyā. This difference is to be added or subtracted to the mean value, as the case may be.

Verse 3 *and first half of* 4. To obtain the time of the conjunction of two planets, compute the difference of the longitudes of two planets, and divide by the difference of their daily motions. If one of the planets be retrograde, divide by the sum of the daily motions. The result gives the number of days approximately after the moment of conjunction if the slower planet has a longitude falling short of that of the quicker. If one of the planets be

retrograde, and if its longitude be the lesser then also the conjunction was past by the number of days computed. In the other cases the conjunction is to take place after the number of days computed. If, however, both the planets be retrograde, then if the slower of them has a longitude less than that of the quicker, then the conjunction is ahead, otherwise past by the number of days.

Comm. Clear. In the case of one or both the planets being retrograde, the word 'slower planet' means that planet whose retrograde motion is slower and not the planet whose mean motion is slower.

Latter half of verse 4 and verse 5. To rectify the moment of conjunction.

Having computed the approximate time of conjunction, obtain the true motions of the planets pertaining to that day, rectify them for Āyana-Dṛk-Karma and following the process indicated in verse (3) above, again compute the moment of conjunction. (This will be a good approximation). This conjunction will be one on the polar latitudinal circle. If Āyana-Dṛk-Karma be not done, then the conjunction will be on the circle of celestial latitude.

Comm. Bhāskara says that the conjunction on the circle of polar latitude is preferred because this could be observed as there is a star at the celestial pole and the movable circle of polar latitude could be moved into a position in which the two planets could be seen situated thereupon. The method of successive approximation is self-explanatory. Bhāskara, however, adds that when the conjunction is on the circle of celestial latitude, the planets will be seen to be closer.

Verse 6 and first half of verse 7. To obtain the north-south celestial latitudinal distance between two planets.

Having computed the number of days by which the celestial latitudinal conjunction was past or is going to take place, let the common celestial longitude of the two planets be obtained by the method of successive approximation for the moment of celestial latitudinal conjunction. Let the celestial latitudes of the two planets be rectified for parallax in latitude as in the case of solar eclipse. The difference of these celestial latitudes in case they are of the same direction or the sum if, of opposite direction, gives the north–south distance of the planets with respect to the ecliptic. Having known the directions of the celestial latitudes with respect to the ecliptic, if the two celestial latitudes happen to be both south or both north, then the planet with lesser celestial latitude is said to be in the opposite direction with respect to the other; that is, suppose both have northern celestial latitudes and suppose p_1 has a smaller northern celestial latitude than p_2, then p_1 is said to be south of p_2. Similar is the case if both the celestial latitudes happen to be south.

Comm. Here Bhāskara does not specify whether he is talking of conjunction on a polar latitudinal circle or on a celestial latitudiaal circle, though the previous procedure indicated by him to obtain the moment of conjunction gives preference to polar latitudinal conjunction which is more easily observable. But, here, as he prescribes rectification of the latitudes for parallax in celestial latitude, we have to construe that he is speaking of conjunction with respect to celestial longitudes alone, because, in the context of parallax, no method was indicated by him for parallzx in polar latitude.

Latter half of verse 7 *and verses* 8 *and* 9. Case of occultation.

If the north–south celestial latitudinal distance happens to be less than the sum of the angular radii of the two planets, an occultation occurs (what we say 'eclipse'

with respect to the Sun and the Moon, holds good with respect to occultation). In this case of occultation, we have to rectify the time of conjunction with respect to parallax in latitude also. For obtaining this parallax in longitude, let the planet which is nearer the earth be taken as the Moon and the other the Sun. But to obtain the longitude of the Vithrībha, which is necessary to compute parallaxes in longitude and latitude, the lagna of the moment of conjunction is to be computed from the position of the Sun and not that of the planet assumed to be the Sun as directed above. Having obtained the parallax in longitude, the computed moment of conjunction is to be rectified for parallax in longitude, (if necessary by the method of successive approximation) to obtain the actual moment of apparent conjunction ie. occultation here. This procedure is worth-adopting only when the occultation in question takes place above the horizon and is observable. The north-south celestial latitudinal distance in this case of occultation corresponds to the celestial latitude of the Moon in the case of a solar eclipse. The direction of this celestial latitude is to be construed as that of the direction in which the planet near the earth is situated with respect to the other. If the planet which is nearer the earth happens to have a motion lesser than that of the other, or be retrograde, then the planet which is situated at a greater distance from the earth will be over taking the other so that the higher planet gets occulted in the eastern direction of its disc. Thus the first contact is to be known to be in the east and the last contact would be in the west; (If otherwise, the other way).

Comm. Self-explanatory.

Here ends the Grahayutyadhikāra.

Bhagrahayuti (Conjunction of a planet with respect to a star)

The longitudes of the stars (professed to be polar).

The polar longitudes of the stars from Aswini including Abhijit are as follows.

	R d m		*R d m*
As'winī	0- 8- 0	Swātī	6-19- 0
Bharaṇī	0-20- 0	Vis'ākhā	7- 2- 5
Krittică	1- 7-18	Anūrādhā	7-14- 5
Rohiṇī	1-19-18	Jyeṣṭhā	7-19- 5
Mṛgas'īrṣa	2- 3- 0	Mulā	8- 1- 0
Ārdrā	2- 7- 0	Purvāṣādhā	8-14- 0
Punarvasū	3- 3- 0	Uttarāṣādhā	8-20- 0
Puṣyamī	3-16- 0	Abhijit	8-25- 0
Asreṣā	3-18- 0	Sravaṇam	9- 8- 0
Makhā	4- 9- 0	Dhaniṣṭhā	9-20- 0
Purvāphalgunī	4-27- 0	S'atabhiṣak	10-20- 0
Uttarāphalgunī	5- 5- 0	Purvābhādrā	10-26- 0
Hasta	5-20- 0	Uttarābhādrā	11- 7- 0
Chitrā	6- 3- 0	Revatī	0- 0- 0

Comm. In the enumeration of these longitudes, Bhāskara makes three statements which we have to note (1) That these longitudes are rectified for Āyana-Dṛk-Karma ie. that they are polar longitudes, (2) That the longitudes of Krittică and Rohinī are less by 32′. (The longitudes shown in the table above are those incorporating this specified correction), (3) That the longitudes of Vis'ākhā, Anūrādhā and Jyeṣṭhā are to be increased by 5′. (This correction is also incorporated in the table given above).

It is to be noted that these longitudes are the same given by Brahmagupta originally and copied by S'rīpati as well as the corrections indicated above. But herein a mistake was committed as clarified by Bhāskara later in the end of the chapter namely that the polar longitudes are subject to what is called Āyanavikāra though the celestial longitudes are not. We say that the celestial longitudes are not subject to such an Āyanavikāra ie. that change due to the phenomenon known as the precession of equinoxes, because the Hindu system is Nirayana ie. reckons the longitudes from the first point of As'winī which is its zero point instead of reckoning from r. If longitudes are measured from r, as r is preceeding, the longitudes of stars will be steadily increasing all at the same annual rate of precession namely about 50–25″ per year. These increasing longitudes of the modern system go by the name Sāyana longitudes. It might be thought that since the polar longitudes also are measured from As'winī and along the ecliptic like celestial longitudes, they also don't vary like the Nirayana celestial longitudes; but it is not so, because the effect of precession on the right ascension and declination of a star have both their effect upon the polar longitude as well as polar latitude. We cannot say that Bhāskara did not know this but we may say that the effect on the polar longitude and latitudes were construed by him as neglible and would be appreciable only in the long run.

Verses 4, 5, 6. Sphutas'aras or rectified latitudes of the stars.

As'winī	10°– 0 north	Ārdrā	11 – 0 south
Bharaṇī	12°– 0 ,,	Punarvasū	6 – 0 north
Krittica	4 –30 ,,	Puṣyamī	0 – 0 ,,
Rohinī	4 –30 south	Asreṣā	7 – 0 south
Mṛgas'irṣa	10 – 0 ,,	Makhā	0 – 0 north

Purvāphalgunī	12 - 0 north	Purvāṣādhā	5 -20 south
Uttarāphalgunī	13 0 „	Uttarāṣādhā	5 - 0 „
Hasta	11 - 0 south	Sravaṇam	30 - 0 north
Chitrā		Abhijit	62 - 0 „
Swātī	37°- 0 north	Dhaniṣṭha	36 - 0 „
Vis'ākhā	1 -20′ south	S'atabhiṣak	0 -20 south
Anūrādhā	1 -45 „	Purvābhādrā	24 - 0 north
Jyeṣṭhā	3 -30 „	Uttarābhādrā	26 - 0 „
Mulā	8 -30 „	Revatī	0 - 0 „

(*a*) These are also what were given by Brahmagupta and copied by S'rīpati, (*b*) Bhāskara mentions in the

सै

" ie. In as much as the stars are fixed we have given their latitudes and longitudes rectified. But here, there is a point to be noted as Bhāskara has put us in a doubt namely that the rectified latitudes which he speaks of elsewhere under verse (3) Grahacchāyādhikāra, Ganitādhyaya are not exactly the polar latitudes spoken of here. (Ref. fig. 120). Let γMR be the ecliptic and γN be the celestial equator. Let S be a star, *k* be the pole of the ecliptic, and *p* be the celestial pole. Then γR is the celestial longitude, SR the celestial latitude which are known as the Dhruvaka and S'ara (Asphuta-s'ara). γM is the polar longitude and SM the polar latitude. MR is the arc connoting the Āyana-Dṛk-Karma correction, so that celestial longitude $\pm$ Āyana-Dṛk-Karma correction = polar longitude (plus or minus according as the longitude lies in the 2nd and 4th quadrants or 1st and 3rd quadrants). If the longitude just equals 0°, 90°, 180° or 270°, the correction of

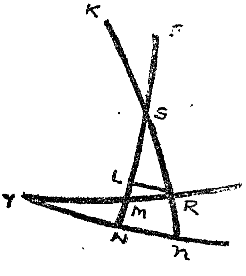


Fig. 120

Karma vanishes. It must be noted here that when the longitude is 0° or 180° Āyanavalana is maximum but Āyana-Dṛk-Karma vanishes. There is no ambignity here in this polar longitude because Bhāskara is unequivocal in defining this. But under verse (3) Grahacchāyādhikāra, Bhāskara means by Sphutas'ara SL and not SM but by Sphutas'ara here he means SM. It is ridiculous to suppose that Bhāskara did not know that $\beta\sqrt{R^2-\text{Āyanavalana}}$ is less than β. This $\beta\sqrt{R^2-a^2}/R$ (a=Āyanavalanajyā) he defined as Sphutas'ara there under verse (3) cited. SR is defined by him as Asphutas'ara or simply S'ara. Now here in the commentary he makes us believe that Sphutas'ara is SM which is the polar latitude. This confusion created by Bhāskara leads Ramaswarup the editor of Brahmagupta Siddhānta as well as one Mukhopadhyaya the author of the thesis, 'Hindu Nakshatras' to suppose that Bhāskara was wrong in supposing that $\frac{\beta}{R}\sqrt{R^2-a^2}>\beta$. Bhāskara could not evidently commit such a silly mistake but we must infer that in that context he called SL as the Sphutas'ara which being added to Rn the Krānti or what is the same LN gives SN the Sphutakrānti or the modern declination. Rn is called by him as Asphutakrānti. In this context he calls SM as the Sphutas'ara since it is Dhruvābhimukha ie. directed towards the pole. Of course, Bhāskara should not have called both SL and SM as Sphutas'aras but since he was deliberately defining the Sphutas'ara as SL previously and now as SM, and since he could not commit such a glaring mistake as to construe $\frac{\beta}{R}\sqrt{R^2-a^2}$ as greater than β, we should not rush to pronounce that Bhāskara was wrong. Only we could say that he is inconsistent to that extent.

7. The polar longitudes of Agastya (Canopus) is 87° and his polar latitude is 77 south. The polar

longitude of Lubdhaka (Sirius) 86° and its polar latitude is 40° south.

Comm. Clear. The star Agastya is considered to be important in Indian literature because the heliacal rising and setting of Agastya are directed to be noted and in fact are being noted in every panchanga even today from times immemorial. Even Kālidāsa alludes to this heliacal rising of Agastya as inaugurating the S'arat-kāla or autumn which was reiterated by Varala Mihira in his Bṛhat-Samhitā under the verse "भगवति जलधरपक्ष्मक्षपाकराकेंक्षणे कमलनाभे, उन्मीलयति तुरङ्गमकरिरथनीराजनं कुर्यात्" ie. when Lord Vishnu whose eyes are supposed to be the Sun and the Moon and his eyelids the clouds, opens his eyes ie. on Kārtica Ekādasi the 11th day of the bright half of the lunar month of Kārtica then Kings are directed to perform Nīrājanavidhi for his horses, elephants and chariots (to start on an expedition for war). It is to be noted that this was so long ago in times of yore that Agastya used to rise at the beginning of autum. Now the star is rising about 22nd August long in advance even in the rainy season. This is on account of the effect of precession of equinoxes.

Verse 8. The Iṣtanādis for Agastya are said to be two, for Lubdhaka $2\frac{1}{6}$, for other stars which are next in size, $2\frac{1}{3}$ and for still smaller ones the Iṣtanādis are to be taken still more.

Comm. Iṣtanādis ie. the time in nādis (where a is equal to 24′ of time) giving the time in between the rising moments of the star and the Sun, when the star rises heliacally. In other words, let the star Agastya rise at a particular time t. If the Sun rises at $t+48'$, then it is the time for Agastya to rise heliacally. This again means that if the Sun sets at time T and the star at T+48′ in its diurnal motion, it is time for the star to set heliacally in the west. Similarly for the other stars. It will be noted here that stars set heliacally in the west and rise

heliacally in the east. This phenomenon has been long in the notice of even the most illiterate people of India from times immemorial, as they were used to get up from beds round about the time when a brilliant star rose heliacally and stood in the eastern horizon, shining for a good length of time before Sun-rise. Each star of first magnitude thus played the part of a morning star for some time, though perhaps the illiterate folk mistake them to be the same star. Especially the brightest stars of the zodiac thus play the part of morning and evening stars. The case with respect to Agastya and Lubdhaka is different in that even though they are far away from the zodiac, yet they were noticed to be morning and evening stars by virtue of their being of the first magnitude at a spot of the sky in the vicinity of which no other such brilliant stars are there. This is the reason why panchanga—Computers have been in the habit of recording in the panchanga even to-date the heliacal rising and setting of Agastya, also because the heliacal rising of this star synchronized with the setting in of S'arat-kāla or autumn. Since in India the lunar Kārtica Ekādasi, ie. the 11th day of the bright half of the lunar month roughly synchronized with 15th Nov., when the Sun rose far in the south of the horizon, this Agastya, though it be in the far south, happens to play the part of a morning star and about the day when it rose heliacally, there were no more rains and waters of the rivers stood crystal-clear as described by many a Sanskrit poet like Kālidāsa (vide the famous verse of Kālidāsa प्रससादोदयादम्भः कुम्भयोने र्महौजसः 4th canto Raghuvams'a).

The star Lubdhaka or Sirius, the dog-star as it is called in English parlance also was conspicuous as a morning and evening star, whose heliacal rising was noticed and recorded by English poets like Shakespeare and Milton.

Since one nādi corresponds to 6°, two nādis correspond to 12°, degrees, which are spoken as Kālāms'as for the

heliacal rising of Agastya. They are so termed, because they indicate the Kāla or the time in between the rising of the star and the Sun which signifies the moment of its heliacal rising. Indirectly therefore these Kālāmsas indicate which star is of which magnitude. A star which has 12° as Kālāmsas is therefore of first magnitude; and as the Kālāmsas increase, the magnitude also increases. It will be rembered that the higher the magnitude of a star, the fainter it will be and not the brighter as is likely to be misconstrued by lay people.

Verse 9. To compute the moment of conjunction of a planet and a star.

The Āyana-Dṛk-Karma is to be done as mentioned before (with respect to the planet) and the Sphutaśara is to be computed to know the time of (polar latitudinal) conjunction.

Comm. We are directed to use the polar longitude and polar latitude with respect to the planet because this kind of conjunction will be more conspicuous than a celestial latitudinal conjunction because the Ecliptic is far more inclined than the Equator with respect to the horizon.

Verses 10, 11.

The difference in the longitudes of the planet and the star, divided by the daily motion of the planet, gives the number of days approximately after or before the moment of conjunction.

If the planet be retrograde, the conjunction past or future will be in the reverse ie. future or past.

Comm. Let x and y be the polar longitudes of the planet and the star and let $x<y$. Since y is constant as the star has no motion, x has to increase to the extent of

y to be in a polar latitudinal conjunction. Hence $(y-x)$ is to be covered by the planet as per its daily motion say m. So the number of days that is to elapse for conjunction is $\frac{y-x}{m}$. If $x>y$, then the conjunction was past by $x-y/m$. If, the planet be retrograde and $x<y$, the conjunction was past by $\frac{y-x}{m}$ days and if $x>y$, the planet being retrograde, the conjunction is to take place in $\frac{x-y}{m}$.

Note. Though Bhāskara does not mention here Asakṛt-Karma ie. method of successive approximation, it is implied because the motion of the planet differs from moment to moment as well as its Sphutaśara. Hence having obtained the approximate moment of conjunction, compute again the true motion of the planet at that instant, as well as its Sphutaśara and Āyana-Dṛk-Karma. The latter ie. Āyana-Dṛk-Karma is to find the polar longitude from the celestial and the Sphutaśara is the polar latitude ie. SM of fig. 120. The Sphutaśara is required for the purpose of finding the distance in between the planet and the star on a polar latitudinal circle and not to compute the moment of conjunction.

Verses 12, 13, 14. To compute the moments of heliacal rising and heliacal setting of a star.

Compute the Udayalagna and the Astalagna of Agastya and Lubdhaka doing Ākṣa-Dṛk-Karma alone. Assuming the Udayalagna to be the Sun, compute the lagna for the Iṣta-kāla nādis (ie. after a lapse of Iṣtanādis after the moment) given before (namely 2 nādis for Agastya and $2\frac{1}{6}$ for Lubdhaka) which will be the longitude of the Sun, when the star (Agastya or Lubdhaka or whatever it be) rises heliacally. Thus the Udayārka is a point of the ecliptic which rises when the star rises heliacally.

Having computed the Asta-lagna of the star, taking it to be the Sun, compute the lagna in the reverse direction ie. the lagna which preceeds it by the Iṣta-kāla given. If this lagna be decreased by 180°, it will give the longitude of the setting Sun at the time of the heliacal setting of the star. Or again find the longitude of the point of the ecliptic which is ahead of the Astalagna which takes (60—Iṣtanādis) to rise after the Astalagna; this longitude decreased by 180° gives the longitude of the setting Sun at the time of the heliacal setting of the star. The heliacal rising or setting takes place when the longitude of the Sun equals the longitude of the point of the ecliptic which is technically called the Udayārka or the Astārka respectively. The difference between the longitude of the Sun and that of the Udayārka or the Astārka divided by the daily motion of the Sun gives approximately the number of days that have elapsed or are to elapse for the heliacal rising or setting as the case may be,

Comm. Here we are to carefully differentiate between the technical words (1) Udayalagna of the star, (2) Asta-lagna of the star, (3) Udayārka and (4) Astārka of the star. The word Udayalagna means that point of the ecliptic which rises simultaneously with the star. The word Astalagna means that point of the ecliptic *which is rising* when the star is setting. The word Udayārka means that point of the ecliptic which rises when the star rises heliacally. The word Astārka means that point of the ecliptic *which is setting* while the star sets heliacally. Thus the four points are only points of the ecliptic.

Let A be the Udayalagna of a star on the circle CAB which is the ecliptic. We know that the position of the star should be above the eastern horizon, to rise heliacally, by such a distance that the time in between the rising of the Udayalagna and that point of the ecliptic which will be rising when the star rises heliacally must be the Iṣta-kāla nādis. Let B be a point ahead of A on the ecliptic

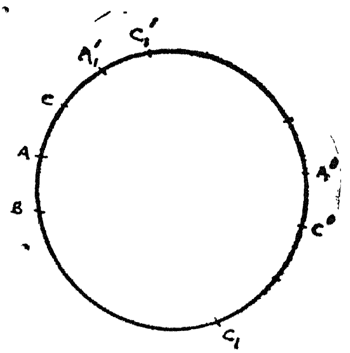


Fig. 121

such that the time in between the rising of B and the rising of A is equal to the Iṣṭa-nādis. By the time B rises, A will have gone up a little above the eastern horizon, as well as the star that has risen along with A and now the position of the star is such that the time in between its rising and the rising of B is equal to the Iṣṭa-nādis. Hence B, which is a point of the ecliptic is termed Udayārka signifying thereby that when the Sun coincides with B, the star rises heliacally. Thus the Udayārka B is ahead of the Udayalagna A and the time in between their risings is equal to Iṣṭa-nādis.

Let now A′ be the Astalagna ie. the point which rises when the star sets. This point will not be, generally, diametrically opposite to A because, the time between the rising and setting of a star which is given by double the rising hour-angle will be far more than half a sidereal day when the star has a large northern latitude and in the case of one having a large southern latitude, the time will be far less than half a sidereal day.

From A′ find C′ which is behind A′ such that the time in between the rising of A′ and C′ is equal to Iṣṭa-nādis. Then the point C diametrically opposite to C′ namely C is called Astārka ie. when the Sun is at C the star sets heliacally in the west. It is so because, when the Sun is on the western horizon setting at C, C′ will be the lagna and when the star is setting A′ is the lagna and as these two lagnas differ by Iṣṭa-nādis, the time between the Sun's setting and the star's setting is also equal to Iṣṭa-nādis.

In other words the star is within the Iṣṭa-nādi distance from the Sun and as C is behind A by Iṣṭa-nādis, the setting Sun at C will be behind (ie. has a lesser longitude) the setting star by Iṣṭa-nādis. Hence the star sets then heliacally. Thus we see that the Udayārka and Astārka given by the points B and C are respectively in advance (ie. has a greater longitude) and behind (ie. has a lesser longitude) of the Udayalagna and Astalagna removed by an Iṣṭa-nādi distance. It will be noted that the arcs AC and AB are not equal though the equatorial arcs corresponding to them are equal.

Instead of finding C′ from A′ by the Vilomalagna method and taking its diametrically opposite point, Bhāskara gives an alternative namely to find C from A′ by the Kramalagna method the time being (60–Iṣṭa-nādikas). This is evident.

Bhāskara prescribes only Ākṣa-Dṛk-Karma to be performed here in finding the Udayalagna and Astalagna of the star because the star's polar longitude is already one in which the Āyana-Dṛk-Karma is contained.

Thus from fig. 121, when the Sun-rises at A, the star rises, when the Suns rises at B the star rises heliacally and when the Sun sets at C, the star sets heliacally.

Verse 15. If in the case of a star, the Astārka happens to have a longitude greater than the Udayārka on account of a very big northern latitude, that star does not set heliacally (and so the question of heliacal rising does not arise).

Comm. In Fig. 121, the Astārka C happens to have a longitude less than that of B (because CAB is the direction of increasing longitude ie. positive direction). At times it so happens that C lies towards the positive direction of B ie. it has a longitude greater than that of B.

This happens when the star has a long northern latitude and therefore the Ākṣa-Dṛk-Karma correction will be sufficiently large and the star has a small north polar distance. This may be substantiated as follows.

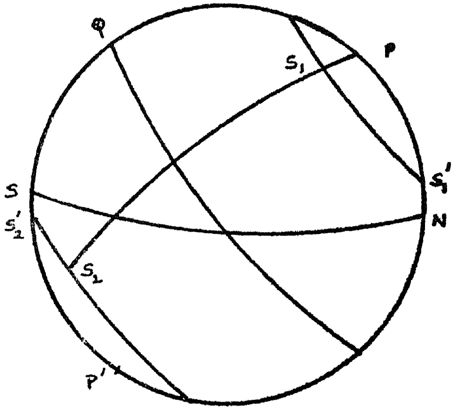


Fig. 122

We have the formula $\cos h = -\tan \phi \tan \delta$ which gives the rising hour-angle of the star. When δ is nearly equal to $90-\phi$, then $\cos h$ will be nearly equal to '-1' which means that the rising hour angle is nearly equal to 180°, ie. the duration of the star's stay below the horizon will be very small. This means A′ approaches A very nearly (fig. 121), for example when it is in the position A_1'. Then let C_1' be the point which is behind A′ by Iṣta-nādis so that C_1' the diametrically opposite point C_1 is far ahead of B in stead of preceeding it. This means that the star which should first set heliacally and then after a few days rise heliacally, is now in such a position that it is to rise heliacally even before setting heliacally. This is an untenable position. That this is not tenable may be seen otherwise. The diametrically opposite point of Astalagna ie. the point C of the ecliptic which should set along with the planet should have a longitude less than that of the

Astārka C_1'; but C_1' will have now a longitude less than C which is incongruous.

In such a situation, Bhāskara says, there is no question of the star setting heliacally at all. This is indeed, an ingenious mathematical presentation of a physical phenomenon, which reflects credit to Bhāskara's genius.

Verse 16. Circumpolar stars or Sadōdita stars.

If a star has a northern declination greater than $90-\phi$ (ie. $\phi > \overline{90-\delta}$) it will be always above the horizon; also if the southern declination is greater than $\overline{90-\phi}$ such a star will never be seen in a northern latitude, be it Lubdhaka or Agastya or even a planet for the matter of that.

Comm. (Vide fig. 122) Let S_1 be a star such that $PS_1' < PN$ ie. $90-\delta < \phi$ ie. $\delta > 90-\phi$. It is clear from the figure that its diurnal path is entirely above the horizon. It is called a Sadōdita star or circumpolar.

Take the case of S_2. Its southern declination ie. QS_2' is greater than the lamba $(90-\phi)$ ie. QS. It is evident from the figure that its diurnal path is entirely below the horizon.

Bhāskara gives two examples here namely (1) where the latitude is greater than 37°, there Agastya will not be visible (having a great north-polar distance), (2) where the latitude is greater than 52°, thēre Abhijit is always above the horizon (having a small north-polar distance).

Bhāskara adds, 'even a planet'. This will be true, for example, in a high latitude say 89°. Suppose the southern declination of a planet is greater than 1°. On that day and for some more days also, the planet's diurnal paths will be below the horizon as will be clear from a figure.

Verses 17 to 21. Ancient astronomers happened to give a list of polar longitudes and polar latitudes at a time when there were no Ayanāmsas ie. when the zero-point of the ecliptic as taken by the Hindu Astronomers namely Aswini coincided with the modern zero-point namely *r*.

In fact these polar longitudes and latitudes do change if there be Ayanāmsas. Here in this case from the polar latitudes the celestial latitudes are to be computed in a reverse process; with the half of these celestial latitudes, the Āyana-Dṛk-Karma is to be effected in the reverse process to obtain the celestial longitudes. After having obtained the correct celestial longitudes and celestial latitudes, now, bringing the Ayanāmsas into the picture, compute the correct Dṛk-Karma and also rectify the celestial longitudes, to obtain the correct polar longitudes and polar latitudes to compute the moment of polar latitudinal conjunction. This case may be taken when the Ayanāmsas are large, otherwise, a small difference there will be, (which does not matter).

Comm. Bhāskara gives the process in the course of the commentary. We have the formula

$$\text{Sphuta Vikṣepa} = \frac{\text{Asphuta Vikṣepa} \times \text{Yaṣti}}{\text{Radius}}$$

Here we know Sphuta Vikṣepa. At once, we could not say $\text{Asphuta Vikṣepa} = \dfrac{\text{Sphuta Vikṣepa} \times \text{Radius}}{\text{Yaṣti}}$ computing Yaṣti from the modern Ayanāmsas. Yaṣti is a function of declination too, because the Āyanavalana is a function of declination. We know, the declinations change in the wake of precession of the equinoxes. So, there is no point in taking the value of the present Yaṣti. We should compute it for the Āyanas'ūnyakāla or for the time when As'wini coincided with *r*. Then the formula could be applied. Then with this celestial latitude obtained, Āyana-Dṛk-Karma is to be effected to obtain the present

polar longitude, using the modern Ayanāmsas. Since in this process, we have no definite knowledge of the then celestial longitude ie., of the Āyanas'ūnyakāla, the method of successive approximation is appealed to.

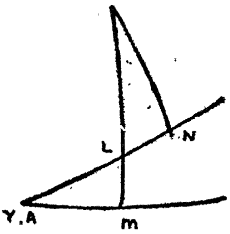


Fig. 123

But this method of Bhāskara may be modified to an easier process as follows. Let rL, LS be the polar longitude and polar latitude of a star as given by Acharyas in whose time the Hindu first point of the zodiac coincided with r. It is required to find rN and NS the celestial longitude and latitude which hold good even today becaus rN and NS are the same as AN, AS, A being the first point of As'wini and a celestial longitude measured from A along the ecliptic is not subject to change on account of precession of the equinoxes as well as the celestial latitude. From the spherical triangle rLM, cot rLM = cos rL tan ω (1) and from the triangle SNL, tan LN = cos $\widehat{SLN}$ × tan SL = cos rLM tan SL (4) and sin SN = sin SL × sin SLN = sin SL × sin rLM (5). From (1) the angle $\widehat{rLM}$ is obtained; substituting this value in (2) and (3) LN and SN are obtained. Adding LN to rL we have rN.

Thus the celestial longitude and latitude are found far more easily and more accurately than from the laborious method indicated which gives only approximate results.

Here ends the Bhagrahayutyadhikāra.

PĀTĀDHYĀYA

Verse 1. Even scholars get confused while computing the occurence of a Pāta, unable to know whether it has already occurred or is going to occur. So, I seek to clarify the method of computing the moment of occurrence of a pāta.

Comm. (1) There are what are called pātās, two in number, called Vyatipāta and Vaidhṛti. The first is defined as occurring at that moment when the Sun and Moon have equal declinations, being situated in opposite Ayanas but the same goḷa. The words Uttaragoḷa and Dakṣinagoḷa are used by Hindu Astronomers as the northern and southern halves of the celestial sphere on either side of the celestial equator, respectively; where as the words Uttara-Ayana and Dakṣiṇa-Ayana are used to connote the times when the Sun or Moon have tropical longitudes (measured from *r* instead of Aswini, the zero point of the Hindu Zodiac) one lying between Capricorn and Cancer, and the other lying between Cancer and Capricorn. Thus Vyatipāta occurs when the Sun and Moon each lies in one of the first or second quadrants of the ecliptic measured from *r* or each lies in one of the third or fourth quadrants of the ecliptic; (both should not lie in the same quadrant) and have equal declinations.

The second Pāta Vaidhṛti occurs when the Sun and Moon have equal declinations, they being situated in the same Ayana but opposite goḷas. These two moments are considered to have malefic effects on humans and the world of life. Though the moments are to be computed by a knowledge of spherical astronomy, they have only an astrological significance. A chapter, usually the last, has been devoted to this subject in every book of Hindu Astronomy. The method of computation was felt difficult

by the ancient Hindu astronomers, before the time of Bhāskara, for the reason that the Moon does not exactly move in the ecliptic but in his own orbit whose inclination to the ecliptic was taken to be $4\frac{1}{2}°$. The points of intersection of the ecliptic with the lunar orbit, or what are called Rāhu and Ketu, the nodes of the lunar orbit, have themselves a motion backwards on the ecliptic in 18.59575 solar years as per Bhāskara. This makes the lunar orbit oscillate about the mean position of the ecliptic, so that sometimes the lunar orbit lies between the celestial equator and some-times not between them. This phenomenon makes the computation more difficult, to obtain the declination of the Moon. Bhāskara says that even great Ācāryas like Lalla and Brahmagupta went wrong or got confused in computing the moments of the occurence of a Pāta. Bearing upon Bhāskara's statement that such Ācāryas also got confused, some second-rate astronomer exclaimed in the following interesting manner "त्रिस्कन्धविद्याकुशलैकमल्लो

भ्रून्त,　　पि किञ्चि

ie. "when even an unrivalled scholar like Lalla, who was well-versed in the three branches of Jyotiṣa betrayed his ignorance in this context of Pātādhikāra, though I am a bit of a mathematician, I could have no pretensions to any knowledge in this difficult chapter".

In Fig. 124, Vyatipāta occurs when SL=MN, where S and M are the Sun and the Moon situated respectively in Uttarāyaṇa and Dakṣiṇāyana but in the same uttaragoḷa. The positions of S and M could be interchanged ie. S may be in the second quadrant of the ecliptic and M in the first quadrant and if their declinations be equal, then also Vyatipāta occurs. Of course in this figure 124, we have taken the Moon also situated on the ecliptic. In

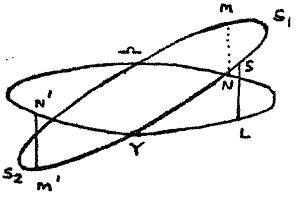


Fig. 124

actual computation, we should not take the Moon as such. The pāta named Vaidhṛti occurs when in the same Fig. 124, $SL=M^1N^1$. Herein both the Sun and Moon are in the same Ayana namely Uttarāyaṇa but in opposite goḷas. If we assume the Moon to be moving on the ecliptic, Vyatipāta occurs when the sum of the tropical longitudes of the Sun and Moon equals 180° and Vaidhṛti occurs when the sum of those longitudes is equal to 360.

Verse 2. Definitions of Goḷa-Sandhi and Ayana-Sandhi with respect to the Sun.

When the tropical longitude of the Sun ie. his longitude measured from ♈ along the ecliptic is equal to 0° or 180°, then he is said to be at his goḷa Sandhi. In other words, when the Sun who moves along the ecliptic comes to the celestial equator, he will be at his Goḷa sandhi. Similarly when his longitude is 90° or 270°, he is said to be at the his Ayana-Sandhi.

Comm. This means that when the Sun is about to pass from the Dakṣiṇa goḷa to the uttaragoḷa or from the uttaragoḷa to the Dakṣiṇagoḷa he is said to be at a Goḷa Sandhi. Similarly when he is about to go south or when he is about to go north, he is said to be at Ayana Sandhi. The word 'Sandhi' means junction. Thus the points ♈ and ♎ (Libra) are said to be goḷa Sandhis whereas the points denoting Cancer and Capricorn are Ayana-Sandhis. Since in Bhāskara's time the Ayanamsas were 11°, ie. the point ♈ was behind the zero point of the Hindu Zodiac by 11°, therefore Bhāskara gives the Goḷa Sandhis as the points having longitudes 349° and 169° respectively. Similarly the Ayana Sandhis were the points having longitudes 79° and 259° respectively.

Bhāskara gives the method of locating these goḷa or Ayana Sandhis as follows. Erect a gnomon

vertically with the help of a plumb-line. Draw a circle on the horizontal plane having the foot of the gnomon as the centre and any arbitrary radius. Draw the East-West and North - South diameters of the circle. The longitudes of the Sun when the shadow of the gnomon lies along the East - West diameter give the goḷa Sandhis. Note the direction of the shadow daily after the day when his shadow is along the East - West line. If the Sun rises thereafter a little towards North of the East point, then he is said to be in the uttara goḷa; his shadow will be a little south of the East - West line. The point at which the Sun thus begins to rise north of the east point, is the Meṣa goḷa Sandhi or the vernal equinox. Gradually the Sun goes on rising at points which are farther and farther away from the East point. When he has reached the extreme north point, ie when the gnomon's shadow is extreme south, he is at the Cancer. Similarly when he rises at the extreme South point on the horizon, he is at Capricorn. Bhāskara actually noted these four points and the longitudes cited above were given by him as the Goḷa Sandhis and Ayana Sandhis. Indirectly, he could know that the Ayanāmśas at the time of his writing the book, were 11°.

Verses 3 to 6. Speciality with respect to the Moon.

Let the Hsine and Hcosine of the tropical longitude of the pāta (Rāhu the ascending node of the lunar orbit) as per the smaller table of Hsines when the radius is taken to be 120′, be respectively multiplied by 123 and 7 and divided respectively by 4 and 12. The results are known as the Bāhuphala and Koṭiphala respectively. According as the tropical longitude of Rāhu ie λ be such that as $270° < \lambda < 90°$ or $90° < \lambda < 270°$, the Bāhuphala is to be divided by $362 \pm$ Kotiphala. Subtract the result from the Goḷa Sandhis and Ayana Sandhis of the Sun, to get those of the Moon.

Comm. from the Hindu Astronomical point of view this is an intricate procedure which made that particular half-learned astronomer declare "पाताधिकारे मम नाऽधिकारः" ie. "I have no pretensions to have understood the chapter known as Pātādhikārā", We perceive herein Bhāskara's mathematical understanding of the problem. His procedure is approximate because he uses (1) the smaller crude table of Hsines taking the radius to be 120′ instead of 3438. (2) also because he gives the Sun's declinations for longitudes 15°, 30° etc. as well as Moon's celestial latitudes for his longitudes of 15°, 30° etc. relative to the node in his orbit. Of course, his mathematical procedure was correct.

He exemplifies his mathematical procedure by solving a problem given in the pras'na–Adhyāya of his book Goḷādhyāya. The problem set by him is as follows.

नांशोऽशशतं शशी

ie. If the tropical longitude of the Moon is 100°, that of the Sun 80°, and the longitude of the Moon measured in his orbit from the Node Rāhu is 200°, compute the moment of the occurence of the pāta, if you know what was said by Lalla in his work S'iṣya dhīvṛddhida. We shall first understand Bhāskaṛa's mind and subsequently we shall a modern procedure.

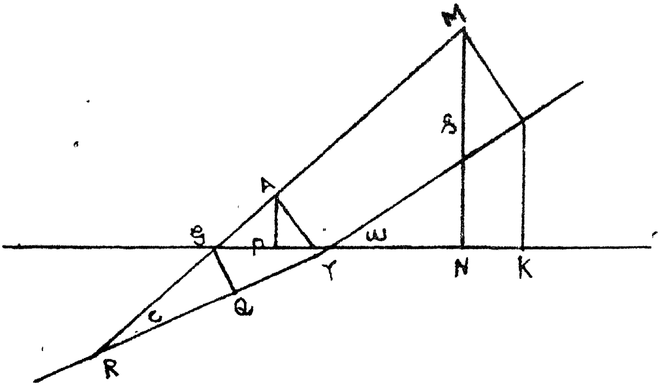


Fig. 125

Refer to fig. 125. Gr NK is the celestial equator. RrL is the ecliptic and RGAM the Moon's orbit where R is the Rāhu or ascending node of the lunar orbit. Let the obliquity of the ecliptic be ω (omega) and the inclination of the lunar orbit to the ecliptic be i. ω was taken to be 24° and i $4\frac{1}{2}$° by Bhāskara. Let the lunar orbit RGAM cut the celestial equator in G which is called the Moon's Goḷa Sandhi. r is the Sun's Goḷa Sandhi. Bhāskara first wants us to locate G ie. to find rQ which gives its longitude. Let A be the position of the Moon when his celestial longitude is zero ie rA is perpendicular to the ecliptic. Let M be the position of the Moon when his celestial longitude is 15° (Here the figure is not drawn to scale but a little exaggerated for the sake of clarity). Let ML be the celestial latitude of the Moon in its position. M. If LK be drawn perpendicular to the equator, LK is called the Asphuta-krānti of the Moon. ML is called the Asphuṭa-Vikṣepa of the Moon. MN drawn perpendicular to the celestial equator ie. the declination of the Moon is called Sphuṭakrānti of the Moon. Draw perpendicular LS on MN. Then MS is called the Sphuṭa Vikṣepa of the Moon. Since MN = Ms + LK Sphuṭakrānti = sphuṭa Vikṣepa + Asphuṭakrānti. Let Ap be the declination of the Moon when his longitude is zero.

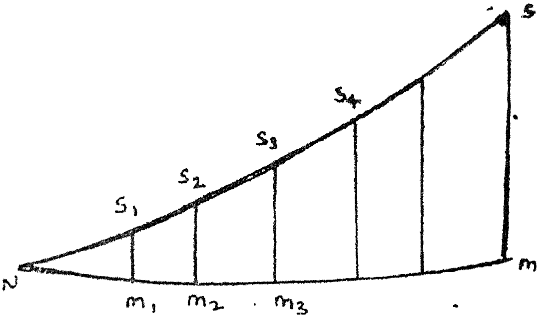


Fig. 126

Now refer to fig. 126. Let NS be the ecliptic and NM the celestial equator. The quadrant NS equal to 90° is divided into six equal parts at S_1, S_2 etc. The successive declinations of S_1, S_2 etc. are given by Brahmagupta as 362, 703, 1002, 1238, 1388, 1440 where the radius is given to be 3438′. In other words 1440 = H Sin ω = H Sin 24°. These can be easily verified from the formula

$$H \text{ Sin } \delta = \lambda \times H$$

putting λ = 15°, 30° etc. and R = 3438′.

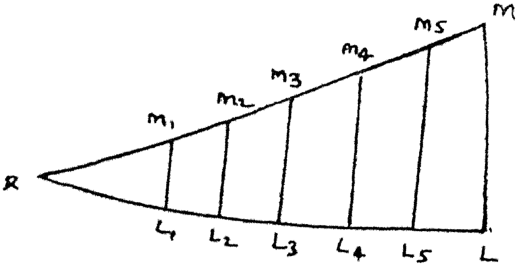


Fig. 127

Now refer to fig. 127. R M is the lunar orbit, R being Rāhu. RL is the ecliptic. Using the formula.

$$H \sin \quad \frac{H \sin \lambda \times H \sin i}{R} \quad \frac{H \sin \lambda \times i}{120}$$

since β and i are small and R is taken to be 120′, we have the successive values of M_1L_1, M_2L_2 etc. of the celestial latitudes of the Moon for arcs RM_1, RM_2 etc. successively equal to 15°, 30° etc. given by 70, 135, 191, 234, 261, 270. In other words the maximum celestial latitude is 270′ = 4½°.

Since when $\lambda = 15°$, taking $R = 120$ $\beta = 70'$, the first S'arakhanda ie the celestial latitude for $\lambda = 15°$, is 70′ $\frac{H \cos \lambda \times 70}{120}$ Bhāskara Calls Koṭiphala.

It will be noted here that the first S'arakhanda is taken to mean the increase in the celestial latitude from zero to 70′ when the longitude increases from 0° to 15°. The argument adduced by Bhāskara in such a context is as follows: "If for a H Cosine λ equal to R equal to 120′ (when $\lambda = 0$) we have the first S'arakhanda namely 70′, what shall we have for an arbitrary H Cosine λ". The result is $\frac{H \cos \lambda \times 70}{120} =$ $\frac{7}{12}$ H cos λ. Since the word Cosine means Koṭijyā, Bhāskara calls this Kotiphala.

Now in fig. 125, let $rL = 15°$, then $LK = 362'$ as given by Brahmagupta. Bhāskara takes this 362 as $\triangle$ (δ) where δ is the declination of the foot of the celestial latitude of the Moon. Then if β be the celestial latitude of the Moon, Bhāskara construes that $\triangle \beta$ is given by the formula $\frac{7}{12}$ H Cos λ, which he calls Kotiphala. Taking $\triangle\delta \pm \triangle\beta$ as the joint variation of δ and β which is roughly equated with the variation in the modern declination MN of the Moon, Bhaskara's argument is "If for a longitude r M = 15° of the Moon corresponds $\triangle\delta \pm \triangle\beta$ what should be the longitude corresponding to the declination Ap of the Moon

when his longitude is equal to zero?" The result is

$$\frac{Ap \times 15}{\Delta\delta \pm \Delta\beta} = \text{II}$$

where rQ is looked upon as the longitude of the Moon at G called the Goḷa Sandhi of the Moon.

But AP = the Sphuṭa Vikṣepa of the Moon when his longitude is equal to zero where rA is the Asphuṭa Vikṣepa at that point. Using his method of calculating AP from Ar, $Ap = \frac{Ar \times H \cos \delta}{R}$ where

$$H \sin \delta = \frac{H \sin \omega \times H \sin (90 + \lambda)}{R}.$$

AP pertains to the longitude λ equal to zero, so that $H \sin \delta = \frac{H \sin 90^\circ \times H \sin \omega}{R} = H \sin \omega.$

$\therefore$ $H \cos \delta = H \cos \omega = H \cos 24^\circ = 110$ when $R = 120'$

Hence $Ap = \frac{Ar \times 110}{120} = \frac{11}{12} Ar.$

But Ar is the celestial latitude of the Moon to be calculated from RA taken to be λ

$\therefore$ $rA = \frac{H \sin \lambda \times 270}{120} = \frac{9}{4} H \sin \lambda$ (where $270' = 4\frac{1}{2}^\circ = i$)

taking $R = 120$ $\therefore$ $Ap = \frac{9}{4} H \sin \lambda \times \frac{11}{12}$. Hence

rQ from I $= \frac{9}{4} H \sin \lambda \times \frac{11}{12} \times \frac{15}{\Delta\delta + \Delta}$

But $\frac{H \sin \lambda}{4} \times \frac{135 \times 11}{12} = \frac{123}{4} H \sin \lambda$. This has to be divided by $\Delta\delta \pm \Delta\beta$. In the problem given. $\lambda = 100^\circ$ and $\Delta\delta$ is taken as 362. To obtain $\Delta\beta$ Bhāskara adduces the argument "If at $\lambda = 0$, the first S'arakhanda of 70 corresponds to $H \cos \lambda = 120'$ what amount of S'arakhanda corresponds to an arbitray $H \cos \lambda$?" The result is $\frac{H \cos \lambda \times 70}{120} = \frac{7}{12} H \cos \lambda$. This he calls Kotiphala because it is based upon $H \cos \lambda$ which is the Kotijyā.

Thus in the given problem where $\lambda = 100°$,

H cos 100° $= \frac{7}{12} \times 21$ (since H cos 100 = − H sin 10 = − 120 × ·1735 = − 21 approximately where R = 120′) = − 12′-15″ ∴ $\Delta\delta \pm \Delta\beta = 362 - 12'\text{-}15 = 349\text{-}45$.

Now $\frac{123}{4}$ H sin λ to be divided above $= \dfrac{123 \times 118}{4}$

$= 3628'\text{-}30''$,

so that $\frac{123}{4}$ H sin $\lambda \times \dfrac{1}{\Delta\delta \pm \Delta\beta} = \dfrac{3628'\text{-}30''}{349\text{-}45}$

$= 10°\text{-}22'\text{-}28'' =$

$= r$Q. Now r's longitude from the zero point of the Hindu zodiac is 11 Rāśis 19° so that Q's longitude = 11 R-19° minus 10°-22′-28″ = 11 Rāśis 8°-37′-32″. This gives us the longitude of G which is called the Moon's Goḷa Sandhi. Adding 3, 6, 9 Rāśis, we get successively the first Āyana Sandhi, the other Goḷa Sandhi and the second Āyana Sandhi of the Moon.

(1) Bhāskara overlooked a crudeness in his procedure namely that $\Delta\beta$ is perpendicular to the ecliptic whereas $\Delta\delta$ is perpendicular to the celestial Equator, but, since $\Delta\beta$ is small, he overlooked the nicety. Strictly speaking $\Delta\beta$ should have been corrected for Āyanavalana.

(2) There is also crudeness in computing or arcs of 15°, whereas they should have been done for increase of every degree in the longitude. He could have done that, because at the end of the Goḷādhyāya he gave the method or computing the H sines for every degree under the caption प्रतिभागज्यकाविधि

We shall now give a modern method of computing the value of rQ. From the spherical triangle RrA, where rR = 100°, $\hat{R} = i = 4\frac{1}{2}°$, we have

$$\sin 100 = \frac{\tan rA}{\tan 4\frac{1}{2}}$$

$\therefore$ log tan rA = log cos 10 + log tan $4\frac{1}{2}°$

= 9·9934 + 8·8960 = 8·8894 $\therefore$ $rA = 4°$–$26'$.

From the spherical triangle rAG,

$$\cos \overline{90-\omega} = \frac{\tan rA}{\tan Gr}$$

log tan Gr = log tan rA –

log sin ω = 8·8894 – 9·6093

= 9·2801. Again from the spherical triangle rQG,

$$\cos \omega = \frac{\tan rQ}{\tan Gr}$$

$\therefore$ log tan rQ = log tan Gr + log cos ω

= 9·2801 + 9·9697 = 9·2498

$\therefore$ $rQ = 10°$–$5'$ whereas Bhāskara got $10°$–$22'$–$28''$.

how Bhāskara was correct mathematically. The small difference there is, due to his taking Rsines for arcs of 15° instead of for smaller arcs.

Note. In the commentary called Sikhā of one Sri Kedāra Datta Joṣi (page 357) we find a mistake committed namely that he subtracted 3 R–10° the longitude of the pāta which is in Rāśis and degrees from the Sphuṭakrānti 345′–45″. What Bhāskara meant was, that since the longitude 100° exceeds 90°, the cosine will be negative which therefore entails $\Delta\beta$ to be subtracted from

Verse 7. How to know the occurrence of a pāta.

If the Sphuṭakrānti of the Moon, when it is be less than that of the Sun, then there could be no occasion for their declinations to become equal in the near future.

Comm. The situation in which the maximum declination of the Moon falls short of that of the Sun, arises

when the lunar orbit comes in between the celestial Equator and the ecliptic. This situation, in its turn, depends upon the position of Rāhu. Since Rāhu's sidereal period amounts to nearly $18\frac{1}{2}$ years, the lunar orbit happens to take its position in between the celestial Equator and the Ecliptic for sufficiently a long period as shown below.

That Bhāskara could visualize this, reflects credit to his genius. This is why he formulated $\Delta\delta \pm \Delta\beta$ stressing upon the plus or minus sign.

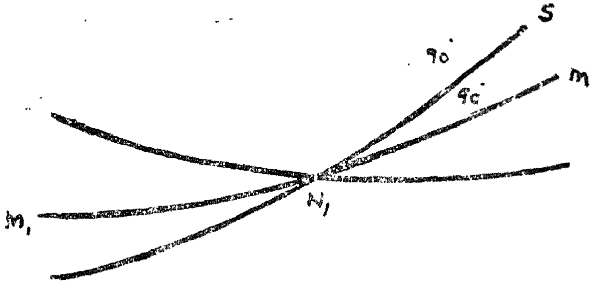


Fig. 128

Bhāskara gives an example under the above verse in the commentary to justify his finding given above. Suppose Rāhu is at the autumnal equinox and the longitudes of both the Sun and the Moon are equal to 79°. Adding the Ayanāms'as 11°, their longitudes are each 90° so that both of them are at the summer solstice ie. an Āyana Sandhi. (Of course the Moon is not exactly at the position of the Sun, but he being in his own orbit, his longitude measured along the Ecliptic is 90°). Then evidently the lunar orbit lies in between the celestial equator and the Ecliptic. Then the declination of the Moon is $24°-4\frac{1}{2}° = 19\frac{1}{2}° = 1170'$ and the declination of the Sun is $24° = 1440'$. Thus the Moon's declination when it is maximum falls short of that of the Sun. After half of the sidereal period of the Moon namely $13\frac{2}{3}$

longitudes of the Sun, Moon and Rāhu are respectively 3-2-28-12, 8-19-4-26 and 6-11-43-28 (in Rās'is, degrees etc), taking mean motions into account. When the Moon has the maximum southern declination in the position M_1 of fig. 129 his longitude will be however 8-10-9-35 and then his declination will be 1169. At that moment the declination of the Sun will be 1398. Even here the Moon's declination falls short of that of the Sun. Again after 13⅝ days, we find the declination of the Moon falling short of that of the Sun. Even after 2 months, the Moon cannot have a declination equalling that of the Sun. This phenomenon occurs when Rāhu of the lunar orbit happens to be at Libra, and the Sun is at the summer solstice approximately.

Verse 8. The definitions of Vyatipāta and Vaidhṛti.

When the Sun and the Moon are in opposite Ayanas, but in the same Goḷa, and if then their declinations be equal, then that moment is said to constitute vyatipātayoga. If on the other hand, if both the Sun and the Moon be in the same Ayana and opposite Goḷas, and if then their declinations be equal, that moment is said to constitute Vaidhṛti Yoga.

Comm. Explained before.

Verse 9 and first half of verse 10. To prognosticate the occurrence of a

When the sum of the tropical longitudes of the Sun the Moon happens to be 180° or 360°, then the Vyatipāta and the Vaidhṛti respectively will occur or will have occured. The number of minutes of arc by which the sum of the tropical longitudes falls short of or exceeds 180° or 360° as the case may be, are to be divided by the sum of the daily motions of the Sun and the Moon, which will aqproximately the number of days after which or

before which the Yogas will occur or would have occured. Compute the declinations of the Sun and the Moon for that moment from the then true daily motions.

Comm. From fig. 124, it is clear that triangle rSL and ≏ MN are congruent so that $r\text{S} = {≏}\,\text{M}$ $\therefore$ $\text{SS}_1 = \text{MS}_1$ $\therefore$ $r\text{S} + r\text{M} = {≏}\,\text{M} + r\text{M} = 180°$. Again triangle rSL and rM$'$N$'$ are congruent $\therefore$ $r\text{S} = \text{M}'r$
$\therefore$ $r\text{S} + r\text{M}' = r\text{M}' + \text{M}'r = 360°$. Thus in the former case which constitutes Vyatipāta, the sum of the tropical longitudes of the Sun and the Moon is 180°; whereas in the latter case, which constitutes the Vaidhṛtipāta, the sum of those two longitudes is equal to 360°.

Hence we are asked to note when the sum of the two longitudes is likely to amount to 180° or 360°. On a particular day suppose the sum is $180-\theta$ or $360-\theta$; so that θ is to be made up by the then velocities of the Sun and the Moon conjointly. Then $\dfrac{\theta}{u+v}$ where u, v are their respective velocities gives the number of days or fraction of a day, by which the sum would be 180° or 360° as the case may be. As the velocities change from moment to moment, the above $\dfrac{\theta}{u+v}$ is only approximate. So, compute again the respective positions and respective velocities. Suppose the sum of the longitudes is $180 \pm \theta'$ or $360 \pm \theta'$ and the velocities u' and v', Then $\dfrac{\theta'}{u'+v}$ gives the fraction of a day after or before the moment in question when the pātas occur Vyatipāta or Vaidhṛti.

Verse 10. To know whether the Yoga is past or future.

If the declination of the Moon situated in an odd quadrant exceeds that of the Sun or falls short of the

Sun's in an even quadraut the moment of the occurence of pāta has elapsed. Otherwise the pāta is to take place shortly after.

Comm. This is clear because in an odd quadrant, the declination of the Moon is on the increase. So if it be already greater than the Sun's, in future it will be far greater and as such cannot equal the Sun's declination. In other words the pāta had already taken place.

Similarly in an even quadrant, the declination of the Moon is on the decrease. As such if it be greater than the Sun's it will become equal in future. Thus the pāta is to take place. On the other hand if the Moon's declination is less than the Sun's, it will decrease further so that prior to the moment concerned a pāta should have occured.

Latter half of verse 11 *and verses* 12, 13, 14. To compute the time of the occurrence of pāta through a consideration of declinations.

Obtain the difference of the declinations if they be of the same direction or their sum if they be of opposite directions in the case of Vyatipāta; obtain the sum or difference of the declinations according as the declinations are of the same or different directions in the case of Vaidhṛti. Call this sum or difference 'the Ādya'. After a lapse of an arbitrary time or before the moment concerned, obtain the positions of the Sun, Moon and the Rāhu; also compute their declinations and form their difference or sum as the case may be. Call this Anya. If on both the occasions it is indicated that the pāta has elapsed or is to elapse, obtain the difference of the Ādya and Anya; otherwise their sum. Divide the arbitrary time taken, by the above sum or difference of the Ādya and Anya and multiply by the Ādya. Take the result in ghatis. Taking this as the arbitrary time, repeat the

process till an invariable quantity is obtained in ghatis. This is the time by which the moment of the pāta has elapsed or after which it is going to occur.

Comm. (1) It might be asked "why bother finding the declinations at all, when it is defined that Vyatipāta happens when the sum of the longitudes of the Sun and the Moon is equal to 180° and Vaidhṛti is defined when the sum is equal to 360, since easily we could know when this happens without taking recourse to declinations?"

The problem is not that simple. The above definition in terms of the sum of the longitudes is an approximate statement, because, the original and correct definition is that their declinations must be equal in magnitude. If they be equal in magnitude but opposite in direction, they constitute Vaidhṛti Yoga. If they be both equal in magnitude and direction, the situation constitutes Vyatipātayoga. In other words, when the diurnal paths of the Sun and the Moon coincide, the moment will be Vyatipāta. If on the other hand their diurnal paths are of equal dimentions but one to the north of the celestial equator and the other to the south, then the moment is Vaidhṛti. Though the sum of the longitudes happens to be 180°, the diurnal paths may not coincide because the Moon does not actually move on the ecliptic. So the situation is more complicated.

Further the calculation of the declination of the Moon is to be done by calculating the Asphutakrānti ie. the declination of the point of the ecliptic which indicates the longitudinal position of the Moon and his celestial latitude ie. the Asphutavikṣepa from the known position of the Node, and then by rectifying the Asphutavikṣepa to obtain the Sphutavikṣepa and then adding the Asphutakrānti to the Sphutavikṣepa to obtain the Sphutakrānti.

(2) *The latter half of verse (11) and the first half of verse* (12).

Let us consider the case of Vyatipāta. Suppose at the moment concerned, the declinations are of opposite direction. Find the sum of their numerical magnitudes. Suppose they are of the same direction; find their difference. This sum or difference of the declinations gives the distance between the planes of the diurnal paths of the Sun and the Moon. This distance is to vanish in order that the diurnal paths may coincide. So we are asked first to find the above distance. Suppose we understand that the Vyatipātha has elapsed by noting the declinations. This may be known easily by the criteria given previously. Supposing x and y to be the declinatious of the Moon and the Sun and supposing that x is on the decrease and approaching y, then the Vyatipātha is to occur. But suppose $x < y$ and x is on the decrease, then the Vyatipāta has taken place. Thus knowing whether the Vyatipātha has elapsed or is to occur, after an arbitrary time t, compute the declinations of the Sun and Moon and form their difference or sum as the case may be which gives the distance between the diurnal paths. Let the first distance found be called Ādya and the second distance the Anya. Then the Anya will be less than the Ādya because we have taken a time towards the occurrence of Vyatipāta when the distance is to get nullified. Find the difference of the Ādya and Anya. Then by the proportion "If in time 't' taken in between the moments of the Ādya and Anya, the distance between the diurnal paths is reduced by Ādya − Anya, what time will be taken for the distance Ādya to vanish?" we have the result

$$T = \frac{\bar{A}\text{dya} \times t}{\bar{A}\text{dya} - \text{Anya}}$$ which gives approximately the time that has to elapse for the occurrence of the pāta. This will be approximate. After a lapse of time T from the Ādya moment concerned, again compute the declinations and repeat the process. We arrive at a particular point of time, which gives the moment of occurrence of the pāta.

In the course of the commentary under these verses Bhāskara solves two problems and points out that when there is a pāta according to his exposition, the statements made by Lalla, Brahmagupta S'rīpati and Mādhava all indicate that there would not occur a pāta.

But we feel that Bhāskara read too much into those statements on account of the following. Lalla states

Brahmagupta states

मेषतुलादौ द्विभाः

चेति

and S'rīpati says

हि

In fact all these three statements mean one and the same and are intended to be approximate statements, in the first instance, having ignored the latitude of the Moon. They are just statements like that of Bhāskara himself when he says that there will be a pāta when the sum of the longitudes will be 180° or 360°. The statements purport to say that if the declination of the Moon when it is on the decrease happens to be less than that of the Sun which is on the increase, there could be no pāta. These statements so far as they go, ignoring the latitude of the Moon are perfectly in order. Bhāskara brings in a detailed analysis of two critical examples, to prove that there is a pāta, but which is negated by the rough statements of the

four Ācāryas. Mādhava's statement in his Siddhānta Cūdāmaṇi is as follows which is also in similar terms as those of the other three.

रवेरोजपद्क्रान्तेः चन्द्रयुग्मपदोद्भवा

स्वल्पाचेन्नतयोःक्रान्त्योः साम्यं स्यादन्यथा भवेत्

Verses 15 and 16. To obtain the duration of the pāta.

The semi-sum of the diameters of the Sun and the Moon or what is the same the sum of their angular radii being multiplied by the Spaṣṭaghatis (obtained in the estimate as per the verses 11–14) and divided by the Ādya in that context, gives the beginning and end of the pāta from the moment of the computed time. The process being repeated according to the method of successive approximations we have the correct estimate of the duration of the pāta.

Comm. This is a convention stipulated with respect to the duration of the Pātā. Strictly speaking, when the diurnal paths of the centres of the discs coincide in the case of Vyatipāta, that will be the middle moment of the Vyatipāta. The pāta is said to last for such a time as the distance between the Centres of the discs (north-south distance) is less than $r+p$. This will be clear from a figure. The situation is akin to that of an eclipse.

The time obtained for the occurrence of the pāta under verses (11) to (14) indicates the middle of the duration of the pāta. The duration of the pāta is defined as the time that lasts as long as the declination of the highest point of one disc becomes equal to that of the lowest point of the other beginning from the moment at which the declination of the lowest point of the one becomes equal to that of the highest point of the second. In other words, just like in an eclipse, so long as the distance between the diurnal paths traced by the centres

of the discs is less than the sum of their angular radii, the pāta lasts. Extending the meaning of this to Vaidhṛti also, so long as the numerical difference of the declinations of the Sun and the Moon ignoring their direction happens to fall short of the sum of the angular radii, the pāta lasts.

To obtain this duration of the pāta the argument is "If the sum or difference of the declinations according as they are of opposite or the same direction (which was taken to be Ādya under verses (11) to (14)) was reduced to zero in the time computed that time being known as Spaṣṭaghatis, what time will be taken for a difference of declinations equal to the sum of the angular radii?"

The result is $\frac{(r+p) \times T}{\bar{A}dya}$ where r and p are the angular radii of the Sun and the Moon, T the time calculated formerly known as Spaṣṭaghatis and Ādya is as defined above. The above result gives the duration of the pāta.

Note. An approximate estimate of this duration could be obtained using differentiation. We have

$$\sin \delta = \sin \lambda \sin \omega \text{ so that}$$

$$\cos \delta \, \Delta\delta = \sin \omega \cos \lambda \, \Delta\lambda$$

Let $\Delta\lambda$ be the motion in longitude of the Moon with respect to the Sun per nādi which will be on the average 12′ approximately.

$$\therefore \quad \Delta\delta = \frac{\sin \omega \cos \lambda \times 12}{\cos \delta}$$

If in one nādi, there be a variation in the declination equal to the above, what time will be taken for 16′ + 15′ the sum of the angular radii approximately? The result is

$$\frac{31 \cos \delta}{12 \sin \omega \cos \lambda}$$

When λ approaches 90°, the duration will be very much since $\cos \lambda \to 0$. This is true because the variation in the declination when $\lambda = 90°$, is very slow even for the Moon. The above formula holds good so long as λ does not approach 90°.

Verse 17. It must be deemed that the declinations will be equal so long as the difference in the declinations is less than $r + p$ numerically.

In the case of Vaidhṛti, we are asked to take the sum instead of difference and difference instead of sum in contradistinction. Why? We shall see. We know that Vaidhṛti occurs when the declinotions of the Sun and the Moon are of equal magnitude but of opposite directions subject to the condition that the longitudes are such that their sum is approximately 360°. This condition is stipulated because when the Sun is in the first quadrant and the Moon is in the 3rd quadrant and at the same time their diclinations are equal, the moment does not constitute Vaidhṛiti, because the sum of the longitudes is then less than 90° + 270° ie. less than 360°. Hence, for Vaidhṛiti one of the two celestial bodies must be in the first quadrant and the other in the fourth quadrant, so that the sum of the longitudes could equal 360° and at the same time the declinations could be equal though of opposite sign. Now compute the declinations of the Sun and the Moon at a particular moment. Suppose they are a and b and both are north. After a few ghatis or days compute again their declinations, say c and d, both again being north. Suppose then $c + d < a + b$, then Vaidhṛiti is going to occur. Construing northern declination to be positive and the southern negative, for Vaidhṛti to occur $a + b$ must be reduced to zero ie. a and b are equal and of opposite direction. Now the argument adduced is "If in time t ghatis or days $\overline{a+b}$ has become $\overline{c+d}$ (calling $a+b$ as Ādya, and $\overline{c+d}$ as Anya) there is a decrease of Ādya - Anya, time should lapse for $\overline{a+b}$ or Adya to reduce to

zero?" The answer is $\frac{t\,(a+b)}{\text{Adya - Anya}}$ which gives the time after which Vaidhṛti is likely to occur.

Suppose a and b the declinations observed at a particular moment, are one north and and the other south. Compute their difference $a-b$; after a few ghatis or days again from the difference of the declinations c and d ie. $c-d$. "If $c-d < a-b$ then Vaidhṛti is going to occur, otherwise it has elapsed. This is so because the difference has to vanish for Vaidhṛti to occur.

Viewing the situation algebraically, ie. construing northern declination to be positive and southern negative, we could have laid down only one criterion, instead of separating the issues and stipulating separate criteria. Proceeding on this basis, suppose the the Sun and the Moon are one in the first quadrant and the other in the second. Here both declinations are therefore positive. Compute their difference $a-b$; compute again the difference of c and d which are again northern declinations after a few ghatis or days. If $c-d < a-b$ is going to occur.

Now take a case when one of a and b is north and the other south; so also c and d. We are talking of a, c to be in the first quadrant and b, d in the fourth quadrant. Even now if $c-d < a-b$ Vyatipāta is going to occur. Thus there is no need to stipulate "Sum or difference". Difference alone counts now, because difference of one positive and the other negative quantities means sum only. Thus in the case of Vyatipāta difference should tend to zero, whether both the declinations are north, or one north and the other south.

Similarly in the case of Vaidhṛti the sum should tend to zero because the algebraic sum of one positive northern declination and the other negative southern declination

shall be zero, if the declinations were to be equal and of opposite directions.

Here ends the Pātādhyāya.

Note Bhāskara gives two examples in the course of the commentary, one, to show the fallacy in asserting that there would be a pāta when the Sun is in an even quadrant, the Moon in an odd quadrant and the Moon's declination is less than that of the Sun; and the other to show the fallacy in asserting that the pāta has elapsed when it is still to take place, and that it is still to occur when it has already elapsed. He quotes the example given in his Golādhyāya wherein the longitudes of the Sun, Moon and Rāhu are 120°, 60° and 180° respectively, there being no Ayamas'as. (Since at the time of Lalla, there were no Ayanams'as, Bhāskara gives such an example), Since the Sun is in an even quadrant and the Moon in an odd quadrant, Bhāskara says, that as per the verse of Lalla 'सूर्यापमात् ओऽपद्दोद्भवात् etc.' there should be a pāta. But since the longitude of Rāhu is 180°, Ketu is at the equinoctial point so that the lunar orbit lies between the ecliptic and the equator. Hence the declination of the Moon remains smaller than that of the Sun for a good length of time never equalling the declination of the Sun. This, Bhāskara says, is a fallacy regarding 'भावाभावत्व' ie. 'Is there a pāta or not' because when there is actually no pāta, Lalla's criterion leads to assert that there is one. It may be reiterated here, that Lalla gave a general criterion ignoring the latitude of the Moon ie. supposing the Moon to be moving on the ecliptic. Bhāskara takes the latitude also into consideration and proves that there is no pāta.

In the second example, which Bhāskara gives, he tries to show the fallacy with respect to "गतैष्यत्व" ie. asserting that a pāta has elapsed when it is still to occur, and that it is to occur when it has already elapsed.

In this example the declinations of the Sun and Moon are respectively 1416′, 1324′, when their tropical longitudes are 80° and 100°. The Moon is in an even quadrant, and his declination is less than that of the Sun. As per the criterion of Lalla given in the verse "सूर्यापमात् etc." the pāta is not there at all. But Bhāskara computes and shows that the pāta has occured 70 nādis before (Rāhu having a reverse tropical longitude of 100°). We shall show here, how using differential calculus we could cut short the process. We have

$$\sin \delta = \sin \lambda \sin \omega \text{; differentiating}$$

$$\cos \delta \, \Delta\delta = \cos \lambda \, \delta\lambda \sin \omega \text{ or}$$

$$\Delta\delta = \frac{\sin \omega \cos \lambda}{\cos \delta} \Delta\lambda$$

$$= \bar{\text{A}}\text{yanavalana} \times \text{Variation in Bhuja.}$$

This could be seen graphically also from fig. 129. Let P and Q be two contiguous positions of the Sun (say) on the ecliptic when arc $pQ = \Delta\lambda$. Let the increment in his

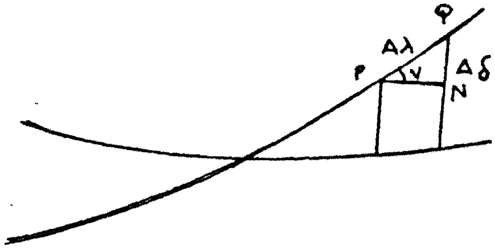


Fig. 129

declination in going from P to Q be NQ equal to Taking PNQ to be a plane △ $\Delta\delta = \Delta\lambda \sin v$ where $v =$ QPN. We have named this angle to be v, because it is no other than the Āyanavalana which was formulated to be equal to $\frac{\sin \omega \cos \lambda}{\cos \; -}$. Thus $\Delta\delta = \Delta\lambda \times$ valanajyā.

APPENDIX

LIST OF TECHNICAL TERMS

1. *Adhikamasa* : A month gained by the lunar reckoning over the solar. This is located in that lunar month which does not contain a

2. *Agrajya* : The Hindu sine of the arc of the horizon in between the rising point of the Sun and the east point.

3. *Akṣa (or) Pala* : Latitude (Terrestrial)

4. *Akṣa Dṛkkarma* : The arc of the ecliptic between the point of intersection of the ecliptic with a secondary through the star to the prime vertical and the point of intersection of the ecliptic with the star's declination circle.

5. *Akṣakarṇa* : The hypotenuse of the gnomonic triangle when its shadow is equal to what is called

6. *Akṣa-Kṣetra* : A right-angled spherical triangle one of whose sides may be the arc of a small circle namely the diurnal circle but one of whose angles is a right angle and another equal to the terrestrial latitude.

7. *Akṣavalanam* : The angle at the point of the star in between the declination circle of the star and a secondary to the prime vertical through the star.

8. *Antyā* : The Hindu sine of an arc of the celestial equator corresponding to *Hṛti*.

9 *Asta* : Setting or heliacal setting.

10. *Ayanabindu* : Solstice.

11. *Āyana-Dṛkkarma*: The arc of the ecliptic intercepted between its point of intersection with the star's declination circle and the secondary to the ecliptic through the star.

12. *Ayanaṁśam* The arc of the ecliptic in between the vernal equinoctial point and the Hindu zero of the ecliptic ie the first point of the Zodiacal sign called *Aśvinī*.

13. *Āyanavalanam* The angle at the point of a star, between its declination circle and the secondary to the ecliptic through the star.

14. *Bārhaspatyamāna*: The time taken by Jupiter to reside in a *Rāśi*, on the average, is called a jovian year. This falls short of a solar year.

15. *Bhāga* : A degree

16. *Cāpa or Kārmuka* : Arc.

17. *Carajyā* : The Hindu sine of the arc intercepted between the east point and the declination circle of a rising star or planet or the Sun.

18. *Cāndra-māsa* : The time between two consecutive full-moons or New moons.

19. *Chāyā or Bhā* : Shadow cast by the gnomon.

20. *Chāyābhuja* : The projection of the shadow on the east-west line.

21. *Chāyākarṇa or Bhākarṇa* : The hypotenuse of the gnomonic triangle whose two sides are the gnomon and its shadow.

22. *Chāyākoṭi*	:	The perpendicular from the extremity of a shadow on the east–west line.
23. *Dhruva*	:	The star near the celestial pole or the celestial pole itself.
24. *Dhruvaka*	:	The celestial longitude.
25. *Dhruva-protavṛttam*	:	The declination circle.
26.	:	The Hindu sine of the azimuth measured by the angle between the prime vertical and the vertical of a star or a planet.
27. *Dorjyā or Bhujajyā*	:	Hindu sine of celestial longitude.
28. *Dṛgjyā*	:	The Hindu sine of the Zenith distance.
29. *Dṛg-lambana*	:	Total parallax.
.0. *Dvāparayuga*	:	Twice the period of a kaliyuga.
31. *Dyujya*	:	The Hindu cosine of declination or the radius of the diurnal circle taking the radius of the celestial equator to be R equal to 3438 units.
32. *Dyujyā-vṛtta or Ahorātra-vṛtta*	:	The diurnal circle of a star or a planet.
33. *Ghaṭi or Nāḍi*	:	An interval of time equal to 24.' (minutes)
34 *Grahaṇa*	:	Eclipse.
35. *Hṛti or Iṣṭahṛti*	:	The Hindu sine of the arc of the diurnal circle from a point of the same upto the plane of the horizon.
36. *Kadamba*	:	Pole of the ecliptic.
37. *Kadamba-protavṛtta*	:	A secondary to the ecliptic through a star or planet.

38. *Kakṣāmaṇḍala* : The deferent of a planet or the circle with the earth as centre and radius equal to 3438 units.

39. *Kaḷā* : The Hindu sine in the diurnal circle corresponding to the *Sūtra* (given below).

40 *Kalā or Liptā* : A minute of angle.

41. *Kaliyuga* : The period consisting of 4,32,000 mean solar years.

42. *Kalpa* : *Dvāparayuga* is twice *Kaliyuga*; *Tretāyuga* thrice and *Kṛta* four times. All these put together constitute a *Mahāyuga*. 71 *Mahāyugas* make one *Manvantara*. Fourteen Manvantaras with what are called Sandhi periods on either side equal to a *Kṛtayuga* or thousand *Mahāyugas* make a *Kalpa*. The creation is supposed to last one *Kalpa*, which is said to constitute the day time of *Brahma*, the Creator. His night also is of the same duration when there is no creation. It is held that we are now in what is called *Śveta-Varāha Kalpa*, wherein the seventh Manvantara called *Vaivasvata manvantara* is current. In this *Manvantara*, twenty seven *Mahāyugas* are supposed to have elapsed and in the twenty-eighth *Mahāyuga*, *kṛta Treta* and *Dvāparayugas* have elapsed and that we are now in the *Kaliyuga*. In this *Kaliyuga* which began on the day when Lord *Kṛṣṇa* gave up his mortal coil, 5080 mean solar years have elapsed by about 21 March, 1979.

43. *Kramajyā or simply jyā or Jīvā or guṇa* : Hindu sine of an angle.

44. *Karaṇa* : Half of the duration of a *tithi*.

45. *Karṇāgrajyā or simply Karṇāgrā*	:	The Hindu sine *Agrajyā* multiplied by K and divided by R where K is the hypotenuse of the gnomonic triangle whose two sides are the gnomon and its shadow and R the radius of the celestial sphere taken to be 3438 units.
46. *Kendra*	:	The centre of a circle.
47. *Ketu*	:	The diametrically opposite point of *Rāhu. Rāhu* also means the circular section of the earth's shadow at the Moon.
48. *Kha-Svastika*	:	The Zenith at a place.
49. *Koṭijyā*	:	Hindu Cosine of an angle.
50. *Krānti or Apama*	:	The declination of a point on the ecliptic.
51. *Krānti-Vṛttam*	:	Ecliptic.
52. *Kṛtayuga*	:	Four times the period of a *Kaliyuga.*
53. *Kṣayamāsa*	:	That lunar month in which there are two *Saṁkramaṇas.*
54. *Kṣitija*	:	The Horizon at a place.
55. *Kujyā*	:	The Hindu sine measured in the diurnal circle corresponding to the *Carajyā* defined above.
56. *Lagna*	:	The *Rāśi* which rises at any moment or the rising point of the ecliptic.
57. *Laṁbajyā*	:	The Hindu Cosine of the latitude.
58. *Laṁbana*	:	Parallax in longitude.
59. *Mahāyuga*	:	The Sum of the four *yugas* mentioned above.
60. *Manvantara*	:	A period equal to 71 *Mahāyugas.*

1. *Nakṣatra*	:	A star. Also the time which elapses as the longitude of the Moon increases by $13\frac{1}{3}$ degrees starting from the zero point of *Aśvinī*.
62. *Nākṣatra-Dina*	:	The time that elapses between two consecutive risings of a star.
63. *Nākṣatra-māsa*	:	The time taken by the Moon to go from *Aśvinī* again to *Aśvinī*.
64. *Natakāla*	:	Hour angle measured in *ghaṭis*.
65. *Natāṁśajyā*	:	Hindu sine of the Zenith distance.
66. *Nati*	:	Parallax in latitude.
67. *Nīcoccavṛtta*	:	Epicycle.
68. *Parama-Antyā*	:	The *Antyā* when the celestial body is at the point of intersection of the celestial equator and the meridian or what is the same at the point of culmination.
69. *Paramakrānti*	:	Obliquity of the ecliptic taken by the Hindu astronomers to be 24°.
70.	:	The point of time when the declinations of the Sun and the Moon are equal and of the same sign or opposite sign. Also it means the point of intersection of two great circles.
71. *Prācī*	:	East point.
72. *Prācyaparā*	:	East-west line.
73. *Prativṛtta*	:	The circle in which the planet moves and whose centre is at a distance from the centre of the Kakṣāmandala or the orbit of the mean planet. This is the same as the orbit of the the true planet supposed to be a circle.

Rāhu	:	The point of intersection of the Moon's path with the ecliptic (ascending point of the Moon's path). Also it means the circular section of the earth's shadow at the Moon.
75. *Rāśi*	:	An arc equal to 30° (on the ecliptic).
76. *Samabindu or Udagbindu*	:	North point.
77. *Samamaṇḍala*	:	Prime vertical.
78. -*Śaṅku*	:	The Hindu cosine of the altitude when the Sun is on the Prime Vertical.
79.	:	The point of time when the Sun enters from one *Rāśi* to another of the twelve *Raśis, Meṣa* etc. measured from the zero-point of the Hindu Zodiac.
80. *Śaṅku*	:	Gnomon.
81. *Śaṅku (Also means)*	:	The Hindu sine of the Altitude.
82. *Śaṅku-cchāyā*	:	The shadow cast by the gnomon.
3. *Saura-māsa*	:	The time when the Sun occupies one *Rāśi*.
84.	:	The Hindu versed-sine of the hour-angle. Also it means celestial latitude.
85. *Sāvanāha*	:	The time between two consecutive Sun-rises.
86. *Sūtra*	:	The Hindu sine of the complement of the hour-angle.
87. *Taddhṛti*	:	The *Hṛti* of a celestial body when it is on the prime vertical.
88. *Tithi*	:	The time taken by the elongation of the moon to increase by 12° starting from zero.

89. *Tithi-kṣaya* — Since a *Tithi* falls short of a Sāvanāha or civil day, in course of time, it so happens, that a tithi begins after sun-rise and does not last upto the next suu-rise. Such a tithi is said to be lost and goes by the name *Tithi kṣaya.* One such tithi kṣaya occurs out of 64 tithis approximately or eleven out of 703 more approximately.

90. *Tretāyuga* : Thrice the period of a *Kaliyuga.*

91. *Trijyā* : The Hindu sine of three Rasis or 90° equal to R or 3438 units.

92. *Udaya* : Rising or heliacal rising.

93. *Unmaṇḍala or Udvṛtta* : The great circle through the celestial pole and the east point.

94. *Unmaṇḍala Śaṅku* : The Hindu cosine of the altitude of a celestial body situated on the *unmaṇḍala.*

95. *Upavṛtta* : A small circle parallel to the prime vertical through a star or a planet.

96. *Vighaṭi or Vināḍi*: One sixtieth of a *ghaṭi.*

97. *Vikṣepa or śara or Viśikha* : Celestial latitude.

98. *Viṣuvatbindu* : Equinoctial point.

99. *Viṣuvat-chāyā* : The shadow cast by the gnomon when the Sun is at the point of intersection of the celestial equator and the meridian on an equinoctrial day.

100. *Viṣuvat-Vṛtta* : Celestial equator.

101. *Vṛtta or Maṇḍala*: A circle.

102. *Yāṁyottara-saṁku or* The Hindu sine of the Altitude of a Celestial body situated on the meridian.

103. *Yāmyottaravṛtta* : The meridian at a place.

104. *Yaṣṭi* $R^2 - Ayanavalanajā^2$

Yaṣṭi has also another meaning namely the length of the perpendicular from a point on the diurnal circle on the plane parallel to the plane of the horizon through the point of intersection of the diurnal circle with the *unmaṇḍala.*

105. *Yoga* The time which elapses when the sum of the longitudes of the Sun and the Moon to increase by $13\frac{1}{3}^{\circ}$ starting from zero.

106. *Yuti* : Conjunction.